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A crash course in basic single-scan target tracking (abridged)

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What is Target Tracking?

Different Impressions Obtained from the Literature:

- ▶ A control systems problem to point an antenna towards an object of interest.
- ▶ The prediction of the future state of a dynamical system based on measurements and models.
- ▶ The act of connecting a vehicle's consecutive positions over time.
- ▶ A problem that was solved by Rudolph E. Kálmán in 1960.

What is Target Tracking?

Target Tracking Is:

- ▶ An aid to reduce the workload of radar operators.
- ▶ A process of finding objects of interest where humans couldn't discern them.
- ▶ An optional part of a radar/sonar system.
- ▶ An indispensable part of a radar system.
- ▶ A critical part of a missile control system or of a counter-missile system.
- ▶ A trivial connecting of the dots.
- ▶ Something that people can do better than the computer.
- ▶ Something that the computer can do better than people.

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The difficulty and utility of target tracking methods depend on the application.

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What Areas of Study Are Relevant?

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What Areas of Study Are Relevant?

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Resources

- ▶ Getting started can be difficult.
- ▶ No comprehensive textbooks on tracking exist.
- ▶ Some useful books:
 - ▶ (Bar-Shalom, Li, and Kriharajan): *Estimation with Applications to Tracking and Navigation: Theory Algorithms and Software*
 - ▶ (Crassidis, Junkins) *Optimal Estimation of Dynamic Systems*
 - ▶ (Bar-Shalom, Willett, Tian) *Tracking and Data Fusion: A Handbook of Algorithms*
 - ▶ (Blackman, Popoli) *Design and Analysis of Modern Tracking Systems*
 - ▶ (Maybeck) *Stochastic Models, Estimation, and Control*
 - ▶ (Stone, Streit, Corwin, Bell) *Bayesian Multiple Target Tracking*
 - ▶ (Challa, Moreland, Musicki, Evans) *Fundamentals of Object Tracking*
 - ▶ (Mahler) *Statistical Multisource-Multitarget Information Fusion*

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- ▶ The International Conference on Information Fusion by the International Society of Information Fusion (ISIF) is the most relevant to target tracking, especially networked/multistatic tracking.
 - ▶ ISIF <http://www.isif.org>
 - ▶ Fusion 2018, Cambridge England: <http://fusion2018.org>
Fusion 2019, Ottawa Canada.
- ▶ The Tracker Component Library (TCL) offers over 1,000 free, commented Matlab routines related to Tracking, Coordinate Systems, Mathematics, Statistics, Combinatorics, Astronomy, etc.
 - ▶ <https://github.com/USNavalResearchLaboratory/TrackerComponentLibrary>
 - ▶ Description of library given in
D. F. Crouse, "The Tracker Component Library: Free Routines for Rapid Prototyping,"
IEEE Aerospace and Electronic Systems Magazine, vol. 32, no. 5, pp. 18-27, May. 2017.

1. Mathematical Preliminaries
2. Coordinate Systems (*in the unabridged slides*)
3. Measurements and Noise
4. Measurement Conversion (*in the unabridged slides*)
5. Bayes' Theorem and the Linear Kalman Filter Update
6. Stochastic Calculus and Linear Dynamic Models (*in the unabridged slides*)
7. The Linear Kalman Filter Prediction
8. Linear Initial State Estimation and the Information Filter
9. Nonlinear Measurement Updates
10. Square Root Filters (*in the unabridged slides*)
11. Direct Filtering Versus Measurement Conversion
12. Data Association
13. Integrated and Cascaded Logic Trackers
14. Dealing with Beams (*in the unabridged slides*)
15. Summary

MATHEMATICAL PRELIMINARIES

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Mathematical Preliminaries

Useful Mathematical Tools

1. Univariate and Multivariate Taylor Series Expansions
 - Given in the unabridged version of the slides.
2. Useful Probability Distributions.
3. Cubature Integration.
4. The Cramér-Rao Lower Bound.

Probability Distributions

- ▶ The four most prevalent probability distributions in target tracking tend to be:
 1. The Multivariate Gaussian Distribution.
 - ▶ Usual assumed noise distribution; discussed in the unabridged slides.
 2. The Central Chi-Square Distribution.
 3. The Binomial Distribution.
 4. The Poisson Distribution.
- ▶ In the TCL, functions relating to these and many other distributions are given in “Mathematical Functions/Statistics/Distributions.”
- ▶ For the above distributions, see GaussianD, ChiSquared, BinomialD, and PoissonD in the TCL.

Probability Distributions: Chi-Squared

- ▶ The central chi-squared distribution with k degrees of freedom is

$$\chi^2(x, d_x) = \frac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} \quad (1)$$
 where Γ is the gamma function.
- ▶ Plotted is $\chi^2(x, 3)$.
- ▶ Confidence regions of a desired % are easily determined using Gaussian approximations, Mahalanobis distances, and chi-squared statistics.

Probability Distributions: Chi-Squared

- ▶ Given a Gaussian PDF estimate of a target, a point $\mathbf{x}$, is within the first p th-percentile if

$$(\hat{\mathbf{x}} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\hat{\mathbf{x}} - \boldsymbol{\mu}) < \gamma_p \quad (2)$$

where γ_p depends on p and on d_x , the dimensionality of x .

d_x	Confidence Region p				
	0.9	0.99	0.999	0.9999	0.99999
1	2.7055	6.6349	10.8276	15.1367	19.5114
2	4.6052	9.2103	13.8155	18.4207	23.0259
3	6.2514	11.3449	16.2662	21.1075	25.9017
6	10.6446	16.8119	22.4577	27.8563	33.1071
9	14.6837	21.6660	27.8772	33.7199	39.3407

Values of γ_p for p and d_x .

- ▶ Use `ChiSquareD.invCDF` in the TCL to determine γ_p .

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Probability Distributions: Chi-Squared

- ▶ The chi-squared distribution plays a role in assessing covariance consistency.
- ▶ The covariance is consistent if it realistically models the error.
- ▶ The Normalized Estimation Error Squared (NEES) is the simplest of multiple methods for assessing consistency.

$$\text{NEES} \triangleq \frac{1}{Nd_x} \sum_{i=1}^N (\hat{\mathbf{x}}_i - \mathbf{x}_i)' \mathbf{P}_i^{-1} (\hat{\mathbf{x}}_i - \mathbf{x}_i) \quad (3)$$

- ▶ $\hat{\mathbf{x}}_i$ and $\mathbf{P}_i$ are estimated mean and covariance from i th random trial.
- ▶ $\mathbf{x}_i$ true value from i th random trial.
- ▶ If estimator is unbiased, covariance is always correct and errors truly Gaussian, then the NEES is $\frac{1}{Nd_x}$ time a central chi-squared random variable with Nd_x degrees of freedom.
- ▶ The function `calcNEES` in the TCL can be useful.

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Probability Distributions: Binomial

- ▶ Consider constant false alarm rate (CFAR) detector with a given P_{FA} per cell, such as the ones given by the CACFAR or OSCFAR functions in the TCL.
- ▶ Grid of N cells (e.g. in range and range-rate).
- ▶ Probability of n false alarms is binomially distributed.

$$\Pr\{n\} = \binom{N}{n} P_{FA}^n (1 - P_{FA})^{N-n} \quad (4)$$

with mean

$$\tilde{\lambda} = NP_{FA} \quad (5)$$

- ▶ The binomial distribution is almost never used in trackers.
- ▶ It is approximated by a Poisson distribution with the same $\tilde{\lambda}$.
 - ▶ The asymptotic equivalence is derived in the unabridged slides.

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Probability Distributions: Poisson

Example:

(a) $N=10$

(b) $N=50$

- ▶ Both plots, $\tilde{\lambda} = 5$ for both distributions.
- ▶ At $N = 1000$, the binomial and Poisson plots look the same.

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Cubature Integration

- ▶ Many integrals cannot be solved analytically with a finite number of terms.
 - ▶ Try to evaluate a Fresnel integral:

$$C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt \quad (6)$$

- ▶ Quadrature integration is a technique for efficient numerical evaluation of univariate integrals.
- ▶ Cubature integration is multivariate quadrature integration.
- ▶ The TCL has many functions related to cubature integration in "Mathematical Functions/Numerical Integration" and "Mathematical Functions/Numerical Integration/Cubature Points."

Cubature Integration: Why?

Numerically integrate the function from 0 to 2.

- ▶ Evaluate

$$\int_0^2 f(x) dx = ? \quad (7)$$

Cubature Integration: Why?

Numerically integrate the function from 0 to 2.

- Basic calculus solution: A Riemann sum:

$$\int_0^2 f(x) dx \approx \sum_{k=0}^{N-1} f(k\Delta_x) \Delta_x \quad \text{where } 2 = N\Delta_x. \quad (8)$$

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Cubature Integration: Theory

- Riemann sums converge very slowly.
- The idea in quadrature/cubature is the relation

$$\int_{\mathbf{x} \in \mathbb{S}} \mathbf{f}(\mathbf{x}) w(\mathbf{x}) d\mathbf{x} = \sum_{i=0}^N \omega_i \mathbf{f}(\mathbf{x}_i), \quad (9)$$

is exact for a particular weighting function w for *all* polynomials f up to a certain order and approximate for other functions f .

- $\mathbb{S}$ is a region, such as $\mathbb{R}^n$ or the surface of a hypersphere.
- Unlike a Riemann sum, N is finite.
- Cubature weights ω_i and points $\mathbf{x}_i$ depend on w and the order.
- Efficient: For a fifth-order integral with a multivariate Gaussian weighting function, one can choose points such that $N = 12$.

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On Solving Integrals

- ▶ Many parts of target tracking involve solving difficult multivariate integrals.
- ▶ Many algorithms fall into one of two categories:
 1. Use cubature integration for the integrals.
 2. Use a Taylor series expansion to turn the problem polynomial and solvable.
- ▶ This comes up again and again.

The CRLB

- ▶ The Cramér-Rao Lower Bound (CRLB) is a lower bound on the variance (or covariance matrix) of an unbiased estimator.
- ▶ The CRLB and a posterior CRLB (PCRLB) are widely used to assess tracker performance.
- ▶ Under certain conditions, the CRLB states

$$\mathbb{E} \{ (\mathbf{x} - \mathbf{T}(\mathbf{z})) (\mathbf{x} - \mathbf{T}(\mathbf{z}))' \} \geq \mathbf{J}^{-1} \quad (10)$$

- ▶ A matrix inequality refers to sorted eigenvalues.
- ▶ $\mathbf{x}$ is the quantity being estimated.
- ▶ $\mathbf{T}(\mathbf{z})$ is the best unbiased estimator.
- ▶ $\mathbf{J}$ is the Fisher information matrix.
- ▶ The expectation is taken over the conditional PDF $p(\mathbf{z}|\mathbf{x})$ if $\mathbf{x}$ is deterministic.
- ▶ The Fisher information matrix has two equivalent formulations:

$$\mathbf{J}^B = -\mathbb{E} \{ \nabla_{\mathbf{x}} \nabla_{\mathbf{x}}' \ln(p(\mathbf{z}|\mathbf{x})) \} \quad (11)$$

$$= \mathbb{E} \{ (\nabla_{\mathbf{x}} \ln(p(\mathbf{z}|\mathbf{x}))) (\nabla_{\mathbf{x}} \ln(p(\mathbf{z}|\mathbf{x})))' \} \quad (12)$$

MEASUREMENTS AND NOISE

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Measurements and Noise

- ▶ Are these points false alarms or a possible track over time?
- ▶ Are they accurate measurements that are far apart?
- ▶ Are false alarms very unlikely or highly likely?

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Measurements and Noise

- ▶ Are these points false alarms or a possible track over time?
- ▶ These are the same points as before at a different scale.
- ▶ Measurements are inherently *noisy*.
- ▶ Knowledge of measurement noise level determines scale.

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Measurements and Noise

- ▶ The blue line is “connect-the-dots.” The orange line just adds interpolation.
- ▶ The blue/orange lines are only good if the points are very accurate.
- ▶ The green line is much more reasonable if the points are inaccurate.
- ▶ The noise level determines the best fit.

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BAYES' THEOREM AND THE LINEAR KALMAN FILTER UPDATE

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Bayes' Theorem

- ▶ Given a PDF $p(\mathbf{x})$ representing the target state estimate at a particular time.
- ▶ Given a measurement $\mathbf{z}$ and a conditional PDF of the measurement $p(\mathbf{z}|\mathbf{x})$.
- ▶ Bayes' theorem states that

$$\underbrace{p(\mathbf{x}|\mathbf{z})}_{\text{posterior distribution}} = \frac{\underbrace{p(\mathbf{z}|\mathbf{x})}_{\text{measurement distribution}} \underbrace{p(\mathbf{x})}_{\text{prior distribution}}}{\underbrace{p(\mathbf{z})}_{\text{normalizing constant}}} \quad (13)$$

- ▶ The value $p(\mathbf{z})$ is essentially a normalizing constant.

$$p(\mathbf{z}) = \int_{\mathbf{x} \in \mathbb{S}} p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) d\mathbf{x} \quad (14)$$

where $\mathbb{S}$ is whatever space $\mathbf{x}$ is in (For discrete variables, the integral becomes a sum).

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Bayes' Theorem and Joint Distributions

- ▶ Bayes' theorem underlies all rigorous measurement update algorithms in tracking.
- ▶ The Kalman filter measurement update is just Bayes' theorem applied to a linear/Gaussian measurement model assuming a Gaussian prior.
- ▶ Notation change for standard tracking:
 - ▶ The "prior" subscript will be replaced by " $k|k-1$ " to indicate that one has an estimate of a current (step k) state given prior (step $k-1$) information.
 - ▶ The "posterior" subscript will be replaced by " $k|k$ " to indicate that one has an estimate of a current state given current information.

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Bayes' Theorem: Linear Gaussian Distributions

- ▶ The discrete measurement update step of the Kalman filter with common notation/terminology.
- ▶ The updated covariance estimate has been reformulated in *Joseph's form* for numerical stability.
- ▶ See KalmanUpdate in "Dynamic Estimation/Measurement Update" in the TCL.

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Bayes' Theorem: Why Use Approximations?

- ▶ The Kalman filter update is optimal for measurements that are linear combinations of the target state.
- ▶ However, why not just apply Bayes' theorem more precisely?
- ▶ Bayes' theorem is again:

$$p(\mathbf{x}|\mathbf{z}) = \frac{p(\mathbf{z}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{z})} \quad (15)$$

- ▶ Just multiply two known functions and normalize the result.
- ▶ Bayes' theorem is trivial. Why not always do it optimally?

Bayes' Theorem: Why Use Approximations?

- ▶ Bayes' theorem is just normalized multiplication. Why not just discretize space and do everything almost optimally on a grid?
- ▶ Simplest "optimal" Bayesian filter:
 1. Discretize the entire estimation space
 2. Evaluate probabilities on a discrete grid for given distributions
 3. Multiply matrices of probabilities to get posterior; normalize
- ▶ It is simple.
- ▶ With parallelization over GPUs, couldn't it be done quickly?

Bayes' Theorem: Why Use Approximations?

- ▶ Why the brute-force grid approach is seldom done:
 - ▶ One target 3D position and velocity in 50 km cube all directions about sensor, speed in any direction to Mach 4 (1372, m/s), discretized to 5 m and 1m/s.
 - ▶ Grid for single probability density function (PDF) is more than 2×10^{22} in size (we need two).
 - ▶ As floating doubles, one grid requires more than 82 zettabytes of RAM (1 ZB=1 trillion GB).
 - ▶ 64GB RAM stick $\approx$ \$255 so cost $\approx$ \$330 trillion (\$660 trillion for two grids, US GDP $\approx$ \$53 trillion).
 - ▶ Computing power to multiply two grids in 1 ms is $\approx$ 20 exaflops.
 - ▶ Most powerful supercomputer (Tianhe-2, China) 33.85 petaflops. We need 612 of them.
- ▶ A smarter approach would be to use some type of adaptive grid or set of points.
 - ▶ This is the basis of particle filters (to be discussed later).
- ▶ **The Kalman filter is much faster than the most efficient particle filter.**

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THE LINEAR KALMAN FILTER PREDICTION

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The Linear Kalman Filter Prediction Summary

- The stochastic dynamic models describe prediction when the initial state $\mathbf{x}$ is deterministic.
- The prediction step of the standard Kalman filter is derived in the unabridged slides and handles random $\mathbf{x}$.
- See the `discKalPred` function in the TCL.

LINEAR INITIAL STATE ESTIMATION AND THE INFORMATION FILTER

Two common approaches to starting the filter are

1. One-point initiation.
 - ▶ See the functions in “Dynamic Models/One-Point Initialization” in the TCL.
2. Using an information filter.
 - ▶ See infoFilterUpdate and infoFilterDiscPred in the TCL.
 - ▶ This is discussed in the unabridged slides.

- ▶ One-point initiation is the simplest approach:
 - ▶ The initial state and covariance matrix are

$$\hat{\mathbf{x}}_{0|0} = \begin{bmatrix} \hat{\mathbf{z}}_{\text{Cart}} \\ \mathbf{0}_{d_x-d_z} \end{bmatrix} \quad (16)$$

$$\hat{\mathbf{P}}_{0|0} = \begin{bmatrix} \mathbf{R}_{\text{Cart}} & \mathbf{0}_{d_z, d_x-d_z} \\ \mathbf{0}_{d_x-d_z, d_z} & \text{diag}([\sigma_1^2, \sigma_2^2, \dots, \sigma_{d_x-d_z}^2]) \end{bmatrix} \quad (17)$$

where

- ▶ d_x and d_z are the dimensionalities of the state and the Cartesian-converted measurement.
- ▶ $\sigma_1^2, \dots, \sigma_{d_x-d_z}^2$ are large variances based on the maximum velocity, acceleration, etc of the target.
- ▶ Known position, other components “uninformative”.
- ▶ Updates and predictions can then be done using the standard Kalman filter.
- ▶ A rule of thumb for σ_i is to use the maximum value of the value of the moment divided by 2 or 3.

NONLINEAR MEASUREMENT UPDATES

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- ▶ Measurement updates are possible without Cartesian conversion.
- ▶ Major nonlinear filtering algorithms shown.
- ▶ We focus on the Extended Kalman Filter and variants of the cubature Kalman filter (which include the “unscented” KF).
- ▶ See EKFUpdate and cubKalUpdate in the TCL.

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Nonlinear Measurement Updates

- ▶ The Kalman filter arose from a Bayesian update given that a linear measurement and the state are jointly Gaussian.
- ▶ Approximating a *nonlinear* measurement

$$\mathbf{z} = \mathbf{h}(\mathbf{x}) + \mathbf{w} \quad (18)$$

where $\mathbf{w}$ is Gaussian, as jointly Gaussian with the state, one still has the same basic update equations as the Kalman filter

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{P}_{k|k-1}^{xx} \left(\mathbf{P}_{k|k-1}^{zz} \right)^{-1} (\mathbf{z} - \hat{\mathbf{z}}_{k|k-1}) \quad (19)$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}^{xx} \left(\mathbf{P}_{k|k-1}^{zz} \right)^{-1} \mathbf{P}_{k|k-1}^{zx} \quad (20)$$

but the quantities $\hat{\mathbf{z}}_{k|k-1}$, $\mathbf{P}_{k|k-1}^{zz}$, $\mathbf{P}_{k|k-1}^{zx}$ are now integrals.

Nonlinear Measurement Updates: CKF

- ▶ The simplest solution to the nonlinear integrals is to use cubature integration, shown above.
- ▶ The square root is a lower-triangular Cholesky decomposition.
- ▶ The vector formulation above requires all cubature weights be positive, but allows for Joseph's form to be used.
- ▶ A Joseph's formulation supporting negative cubature weights is probably impossible.

Nonlinear Measurement Updates: EKF

- ▶ An alternative approach is to use a Taylor series expansion of the nonlinear function.
- ▶ The result is the extended Kalman filter (EKF), shown above.

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DIRECT FILTERING VERSUS MEASUREMENT CONVERSION

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Filtering Versus Measurement Conversion

- ▶ Two common approaches for basic tracking exist:
 1. Cartesian converting measurements (and covariances) and using a linear filter.
 2. Directly using measurements in a nonlinear filter.
- ▶ These shall be compared in a simple example.

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Filtering Versus Measurement Conversion

- ▶ $\mathbf{R}^1 = \text{diag}([10 \text{ m}, 10^{-2}, 10^{-2}])$.

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- ▶ A flat Earth.
- ▶ All ships on the surface traveling -10 m/s in the negative z direction.
- ▶ The target initially at an altitude of 7 km going 100 m/s .
- ▶ Radars on ships pointed 15° up from the horizontal.
- ▶ $\tilde{q} = 0.4802 \text{ m}^2/\text{s}^3$
- ▶ Measurements every $T = 0.5 \text{ s}$.
- ▶ Tracks initialized via an information filter with 2 converted measurements.

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Filtering Versus Measurement Conversion: RMSE

- ▶ The positional RMSE error of three different tracking algorithms. The CKF used 5th order points.
- ▶ The CKF has the best RMSE performance.

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Filtering Versus Measurement Conversion: NEES

- ▶ The NEES of three different tracking algorithms.
- ▶ The EKF is bad; the CKF is the best over time; converted measurements are initially the best.

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DATA ASSOCIATION

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Data Association

- ▶ Common algorithms for assigning measurements to targets shown.
- ▶ We focus on non random finite set (RFS)-based single scan approaches.

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Topics considered are:

1. The Likelihood Function.
2. Naïve Nearest Neighbor, the Score Function, and Global Nearest Neighbor (GNN)
3. Probabilistic Data Association (PDA) and Joint Probabilistic Data Association (JPDA) variants

- ▶ Consider one known target with a Gaussian prediction $\hat{\mathbf{x}}_{k|k-1}$, $\mathbf{P}_{k|k-1}$ with a 100% detection probability and with N_M measurements present.
- ▶ Which measurement should be assigned to the target?
- ▶ Single-scan data association algorithms make this decision based only on the current state prediction $\hat{\mathbf{x}}_{k|k-1}$, $\mathbf{P}_{k|k-1}$.
- ▶ Multiple scan data association look at multiple sets of measurements.

The Likelihood Function

- ▶ This is usually bad:
 - ▶ Measurements are more accurate in range than cross range.
 - ▶ Cross-range becomes worse farther away from sensor, as illustrated (monostatic).
 - ▶ The shape of the uncertainty region of the state can matter.
 - ▶ Target ellipse crosses multiple range cells in image.

$$i = \arg \min_i \left\| \mathbf{H}^p \mathbf{x} - \mathbf{z}_i^{\text{Cart}} \right\|^2 \quad (21)$$

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The Likelihood Function

- ▶ One cannot convert the state to the measurement coordinate system and use a similar l_2 norm.
 - ▶ Mixing units (e.g. range, angle, and even range rate) makes no sense.
- ▶ Valid distance measures can be derived from likelihood functions and likelihood ratios.
 - ▶ Another reason that measurement covariance matrices matter.
- ▶ Let $\mathbf{Z}^{k-1}$ be the set of all measurements up to discrete time $k-1$ and Θ^{k-1} be the information of which measurements are assigned to the track up to time $k-1$.
- ▶ A valid cost function is the likelihood $p(\mathbf{z}|\mathbf{Z}^{k-1}, \Theta^{k-1})$.

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The Likelihood Function

- ▶ Written out, the likelihood of the i th measurement:

$$p(\mathbf{z}_i | \mathbf{Z}^{k-1}, \Theta^{k-1}) \triangleq \tilde{\Lambda}(\theta^i) = \left| 2\pi \mathbf{P}_{k|k-1}^{zz,i} \right|^{-\frac{1}{2}} e^{-\frac{1}{2} (\mathbf{z} - \hat{\mathbf{z}}_{k|k-1})' (\mathbf{P}_{k|k-1}^{zz,i})^{-1} (\mathbf{z} - \hat{\mathbf{z}}_{k|k-1})} \quad (22)$$

- ▶ $\mathbf{P}_{k|k-1}^{zz,i}$ depends on the covariance matrix $\mathbf{R}_i$ of the i th measurement.
- ▶ Taking the negative logarithm of the likelihood and dropping the normalizing constant terms and $1/2$ scale factor one has a *Mahalanobis distance*:

$$-\log(\tilde{\Lambda}(\theta^i)) \propto (\mathbf{z} - \hat{\mathbf{z}}_{k|k-1})' (\mathbf{P}_{k|k-1}^{zz,i})^{-1} (\mathbf{z} - \hat{\mathbf{z}}_{k|k-1}) \quad (23)$$

- ▶ From the mathematics section, we know that Mahalanobis distances can be used for chi-squared testing to determine whether measurements can even be considered valid.
 - ▶ The exclusion of measurements from possible assignments is **gating**.

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Naïve Nearest Neighbor

- ▶ For multiple targets, one is tempted to assign the highest likelihood measurement to each target.
- ▶ In the above scenario, both targets would be assigned to measurement m_1 .
- ▶ Naïve nearest neighbor leads to track coalescence and ultimately, needless track loss.
- ▶ A practical algorithm must assign measurements jointly across targets, accounting for missed detections.
- ▶ Naïve nearest neighbor is one of the options in **singleScanUpdate** in the TCL.

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The Score Function

- ▶ We want to derive a cost function (a score function) that can be used for multiple target assignment.
- ▶ The exponential of the score function derived in the unabridged slides here is computed in `makeStandardLRMatHyps` and `makeStandardCartOnlyLRMatHyps` in the TCL.

The Score Function

- ▶ Under many standard assumptions, the marginal change in the log-likelihood for assigning a measurement is

$$\Delta\Lambda_{t,i} = \begin{cases} \ln\left(P_D^t \frac{\mathcal{N}\{\mathbf{z}_i, \hat{\mathbf{z}}_{k|k-1}^t, \mathbf{P}_{k|k-1}^{zz,i,t}\}}{\lambda}\right) & \text{if } i \neq 0 \\ \ln(1 - P_D^t) & \text{if } i = 0 \end{cases} \quad (24)$$

- ▶ $\hat{\mathbf{z}}_{k|k-1}^t$ is the predicted measurement from the t th target,
- ▶ $\mathbf{P}_{k|k-1}^{zz,i,t}$ is the innovation covariance the for i th measurement and t th target.
- ▶ The term $\Delta\Lambda_{t,i}$ is the marginal *score function* for single-frame assignment.
- ▶ Summing the marginals for a full target-measurement assignment, one forms the full score function $\Lambda(\theta)$ for a scan.

The Score Function

- ▶ When using a converted measurement filter, the units of $\mathcal{N}\{\mathbf{z}_i, \hat{\mathbf{z}}_{k|k-1}^t, \mathbf{P}_{k|k-1}^{zz,i,t}\}$ are in Cartesian coordinates, but the units of λ are usually in the radar's local coordinates.
- ▶ The proper conversion of λ to Cartesian coordinates yields a different λ at every point.
 - ▶ Cartesian λ is higher closer to the sensor.
- ▶ The Cartesian version of λ given λ in the measurement coordinate system is

$$\lambda_x = \frac{1}{|\mathbf{J}(\mathbf{y})|} \lambda_y \quad (25)$$

- ▶ In the TCL, necessary Jacobians are in "Coordinate Systems/Jacobians/Converted Jacobians" and include `calcRuvConvJacob` and `calcPolarConvJacob`, among others.

GNN Assignment

- ▶ One could assign measurements to targets and false alarms by choosing the assignment θ that maximizes the score function.
- ▶ How many valid assignments are there for m measurement and N_T targets?

$$N_{\text{hyp}} = \underbrace{\sum_{l=0}^{\min(m, N_T)}}_{\text{Sum over the number of targets observed}} \underbrace{\binom{N_T}{l}}_{\text{Choose which targets are observed}} \underbrace{\binom{m}{l}}_{\text{Choose which measurements are not false alarms}} \underbrace{l!}_{\text{Assign the measurements to the targets}} \quad (26)$$

- ▶ Suppose there are 3000 measurements and targets, and no false alarms or missed detections.
 - ▶ There are $3000! \approx 4.14 \times 10^{9130}$ hypotheses.
 - ▶ This is about one googol (10^{100}) raised to 91.3.

GNN Assignment

- ▶ There are $3000! \approx 4.14 \times 10^{9130}$ hypotheses, but only $3000^2 = 9 \times 10^6$ marginal hypotheses (values of $\Delta\Lambda_{t,i}$).
- ▶ The efficient solution is formulated as a GNN assignment (2D assignment) problem:

$$\mathbf{x}^* = \arg \max_{\mathbf{x}} \sum_{i=1}^{N_R} \sum_{j=1}^{N_C} \Delta\Lambda_{i,j} x_{i,j} \quad (27)$$

$$\text{subject to } \sum_{j=1}^{N_C} x_{i,j} = 1 \quad \forall i \quad \text{Every target is assigned to an event.} \quad (28)$$

$$\sum_{i=1}^{N_R} x_{i,j} \leq 1 \quad \forall j \quad \text{Not every event is assigned to a target.} \quad (29)$$

$$x_{i,j} \in \{0, 1\} \quad \forall x_{i,j} \quad \text{Equivalent to } x_{i,j} \geq 0 \quad \forall x_{i,j} \quad (30)$$

- ▶ $N_R = N_T$ and $N_C = N_T + m$, number of measurements plus missed detection hypotheses.

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GNN Assignment

- ▶ Each target gets its own missed detection hypotheses; costs for other targets' hypotheses are $-\infty$.
- ▶ To use the algorithm note that the cost matrix takes the form

$$\mathbf{C}_l \triangleq \begin{bmatrix} \overbrace{\Delta\Lambda_{1,1} \dots \Delta\Lambda_{1,m}}^{\text{Assignment Costs}} & \overbrace{\Delta\Lambda_{1,0} \dots -\infty}^{\text{Missed Detection Costs}} \\ \Delta\Lambda_{2,1} \dots \Delta\Lambda_{2,m} & -\infty \Delta\Lambda_{2,0} \dots -\infty \\ \vdots & \vdots \\ \Delta\Lambda_{N_T,1} \dots \Delta\Lambda_{N_T,m} & -\infty \dots \Delta\Lambda_{N_T,0} \end{bmatrix}. \quad (31)$$

- ▶ 2D assignment is a binary integer programming problem.
- ▶ A polynomial time solution is implemented as assign2D and kBest2DAssign in the TCL

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- ▶ The GNN algorithm is a maximum-likelihood approach.
- ▶ An alternative is to use the expected value over all possible target-measurement assignments.
- ▶ For a single target, the expected value and the covariance of the estimate are called *probabilistic data association* (PDA).
- ▶ For multiple targets, it is called Joint Probabilistic Data Association (JPDA).
- ▶ Variants of the PDA and JPDA are implemented in `singleScanUpdate` in the TCL.

- ▶ For the t th target, the JPDA update is

$$\mathbf{x}_{k|k}^t = \mathbb{E} \{ \mathbf{x}_k^t | \mathbf{Z}, I_p \} = \sum_{i=0}^m \beta^{i,t} \hat{\mathbf{x}}_{k|k}^{t,i} \quad (32)$$

$$\mathbf{P}_{k|k}^t = \mathbb{E} \left\{ \left(\mathbf{x}_k^t - \hat{\mathbf{x}}_{k|k}^t \right) \left(\mathbf{x}_k^t - \hat{\mathbf{x}}_{k|k}^t \right)' \middle| \mathbf{Z}, I_p \right\} \quad (33)$$

$$= \sum_{i=0}^m \beta^{i,t} \left(\mathbf{P}_{k|k}^{t,i} + \left(\mathbf{x}_k^{t,i} - \hat{\mathbf{x}}_{k|k}^t \right) \left(\mathbf{x}_k^{t,i} - \hat{\mathbf{x}}_{k|k}^t \right)' \right) \quad (34)$$

- ▶ $\beta_{i,t}$ is the probability of assigning measurement i to target t (0 is a missed detection).
- ▶ Superscripts of i and t indicate measurement and target hypotheses.
- ▶ I_p is information on the (assumed Gaussian) prior estimates.
- ▶ The literature often uses a simpler expression for $\mathbf{P}_{k|k}^t$ that is not quadratic in form and subject to finite precision errors.

The PDA and JPDA Algorithms

- ▶ Assumptions going into the PDA/JPDA are that the prior distributions on all targets are Gaussian.
- ▶ The covariance cross terms between targets are not zero, but are omitted.
- ▶ The hardest part of the PDA/JPDA is the computation of the β values.

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The PDA and JPDA Algorithms

- ▶ Gating and clustering are important parts of a large-scale JPDA implementation.
- ▶ In the above figure, measurements are said to gate with a target if in the ellipse overlaps them.
 - ▶ In practice, use a chi-squared test on the Mahalanobis distance.
- ▶ There are three clusters of targets and measurements.
 1. Target t_1 is in a cluster with m_4 .
 2. Targets t_2 and t_3 (linked by m_2) cluster with m_1 , m_2 , and m_3 .
 3. Target t_4 is in a cluster with m_6 .

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The PDA and JPDA Algorithms: Gating and Clustering

- ▶ Brute-force gating and likelihood evaluation is implemented in the TCL via the makeStandardLRMatHyps and makeStandardCartOnlyLRMatHyps functions.
- ▶ Clustering can be computationally efficiently performed using disjoint sets, an obscure Computer Science data structure.
- ▶ Disjoint sets for clustering are implemented in the DisjointSetM and DisjointSet classes in the TCL; DisjointSet keeps track of only targets in clusters; DisjointSetM keeps track of targets and measurements in clusters.

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The PDA and JPDA Algorithms: Gating and Clustering

- ▶ An illustration of how separate clusters can be processed independently.
- ▶ $\mathbb{G}_g$ is the set of targets and measurements in the g th cluster.

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The PDA and JPDA Algorithms: Computing β

- ▶ When the β terms must be computed exactly, two approaches shall be considered:
 1. Via brute-force evaluation of all joint association events.
 2. Via matrix permanents.
- ▶ The matrix permanent approach is faster, but brute force is necessary to derive some JPDAF variants.

The PDA and JPDA Algorithms: Computing β

- ▶ Consider a matrix of likelihoods with $\Delta\tilde{\Lambda}_{t,i} = e^{\Delta\Lambda_{t,i}}$, non-normalized assignment probabilities:

$$\mathbf{C} \triangleq \begin{bmatrix} \overbrace{\Delta\tilde{\Lambda}_{1,1} \quad \dots \quad \Delta\tilde{\Lambda}_{1,m}}^{\tilde{\mathbf{C}}: \text{Assignment Likelihoods}} & \overbrace{\Delta\tilde{\Lambda}_{1,0} \quad 0 \quad \dots \quad 0}^{\text{Missed Detection Likelihoods}} \\ \Delta\tilde{\Lambda}_{2,1} \quad \dots \quad \Delta\tilde{\Lambda}_{2,m} & 0 \quad \Delta\tilde{\Lambda}_{2,0} \quad \dots \quad 0 \\ \vdots & \vdots \\ \Delta\tilde{\Lambda}_{N_T,1} \quad \dots \quad \Delta\tilde{\Lambda}_{N_T,m} & 0 \quad 0 \quad \dots \quad \Delta\tilde{\Lambda}_{N_T,0} \end{bmatrix} \quad (35)$$

- ▶ The normalized expression for the β terms can be rewritten directly in terms of likelihoods using elements of $\mathbf{C}$:

$$\beta_{j,k} = \Delta\tilde{\Lambda}_{j,k} \frac{\sum_{\sigma \in \mathbb{P}^{N_T-1, N_T-1+m}} \prod_{\substack{n=1 \\ n \neq j}}^{N_T} c_{n, \sigma_n}}{\sum_{\sigma \in \mathbb{P}^{N_T, N_T+m}} \prod_{n=1}^{N_T} c_{n, \sigma_n}} \quad (36)$$

- The expression simplifies to

$$\beta_{j,k} = \Delta \tilde{\Lambda}_{j,k} \frac{\text{perm}(\bar{\mathbf{C}}_{j,k})}{\text{perm}(\mathbf{C})} \quad (37)$$

where $\bar{\mathbf{C}}_{j,k}$ is the matrix $\mathbf{C}$ after removing row j and column k .

- The matrix permanent cannot be evaluated in polynomial time unless $P=NP$.
- Efficient exponential complexity algorithms exist. In the TCL, the function `perm` implements an efficient algorithm.

- Functions to explicitly compute the β values are implemented in the `calc2DAssignmentProbs` function in the TCL.
- Many techniques to approximate β values exist and are implemented in `calc2DAssignmentProbsApprox` in the TCL.
- Methods to do the complete PDA and JPDA update are given in `singleScanUpdate` in the TCL.
- However, one usually uses a variant of the JPDA algorithm rather than the JPDA algorithm itself.

The JPDA Algorithm: Coalescence

- ▶ Consider two targets whose states consist only of scalar position and have been stacked.
- ▶ Suppose that the joint PDF for the two targets is

$$p(\mathbf{x}) = \frac{1}{2} \delta \left(\mathbf{x} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) + \frac{1}{2} \delta \left(\mathbf{x} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) \quad (38)$$

- ▶ One target is located at +1 and one target is located at -1, but we do not know which.
- ▶ $E\{\mathbf{x}\} = \mathbf{0}$, where no target is located.
 - ▶ Identity uncertainty causes track coalescence!
- ▶ Coalescence is not a “bias”.
- ▶ Coalescence is the result of using the expected value given uncertain identity.

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The JPDA Algorithm: Coalescence

- ▶ The Set JPDAF, the GNN-JPDA and the JPDA* can reduce coalescence.
- ▶ The GNN-JPDA is simple:
 1. Determine the measurement to use with a GNN filter, giving $\hat{\mathbf{x}}_{k|k}$.
 2. Compute $\mathbf{P}_{k|k}$ as in the JPDA, using the GNN estimate as the mean $\hat{\mathbf{x}}_{k|k}$.
- ▶ The hard assignment avoids coalescence.
- ▶ Computing $\mathbf{P}_{k|k}$ as a MSE matrix improves covariance consistency/reduces track loss.
- ▶ Available as an option in `singleScanUpdate` in the TCL with exact and approximate β s.

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The JPDA Algorithm: Coalescence

- ▶ The brute-force computation of the β s had loops:
 1. Choose how many targets are observed.
 2. Choose which targets are observed.
 3. Choose which measurements originated from targets.
 4. Permute all associations of observed targets to target-originated measurements.
- ▶ The JPDA* is the same as the JPDA except in the innermost loop, only the maximum likelihood permutation is used.
 - ▶ Has the smoothing of the expected value.
 - ▶ The hard decision gets rid of identity uncertainty: Resistant to coalescence.
- ▶ Use calcStarBetasBF for the β s in the TCL. Available as an option in singleScanUpdate in the TCL.

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The JPDA Algorithm: Example

- ▶ A 2D example of the JPDA* including gating and clustering is given in demo2DDataAssociation in "Sample Code/Basic Tracking Example" in the TCL.
- ▶ A sample run is shown above. Tracks were started from two cued measurements.
- ▶ Estimated tracks: Red. True track: Dashed black. Detections: Blue. Very resistant to false alarms.

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CASCADED LOGIC AND INTEGRATED TRACKERS

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Cascaded Logic and Integrated Trackers

- ▶ The GNN and JPDA algorithms only update established tracks.
- ▶ Most practical systems require the ability to start and terminate tracks.
- ▶ Two main categories of algorithms exist for single-scan data association approaches:
 - ▶ Cascaded Logic Trackers
 - ▶ Confirmed-tracks, pre-tracks and hard decisions for initiation and termination.
 - ▶ Integrated Trackers
 - ▶ Lots of targets, each with a probability of existing.

A Cascaded Logic Tracker

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- ▶ Multiple Types of cascaded logic trackers exist.
- ▶ There are confirmed tracks and candidate tracks.
 - ▶ Sometimes pre-tracks too.
- ▶ Scores usually updated via GNN assignments.
- ▶ Measurements not in GNN assignments go on to the next stage.
- ▶ Creation, promotion and deletion of tracks in purple-outlined boxes.

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An Integrated Tracker

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- ▶ Integrated trackers maintain a probability of target existence with each possible target.
- ▶ Usually, a track is not considered firm until its existence probability exceeds a threshold.
- ▶ A track is not terminated until its existence probability goes below a lower threshold.
- ▶ Measurement update implemented in the singleScanUpdateWithExistence function in the TCL.

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An Integrated Tracker

- ▶ A rigorous derivation of the JIPDA class of filters is usually done using finite set statistics.
- ▶ A proper coverage of finite set statistics is beyond the scope of this presentation.

- ▶ An example of a minimal end-to-end GNN-JIPDAF in 2D is given in `demo2DIntegratedDataAssociation` in "Sample Code/Basic Tracking Examples" in the TCL.
- ▶ A plot of a run of the sample code with the detections and found tracks (green) and true tracks (red) is shown above for the simple two-target scenario.

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SUMMARY

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Summary I

- ▶ Gaussian approximations and Poisson clutter are widely used.
- ▶ Tracking algorithms need consistent measurement covariance matrices. Cross terms between range and range rate can matter.
- ▶ The Kalman filter comes from a Bayesian update of a linear dynamic model and a linear measurement.
- ▶ The EKF and CKF use Taylor series and cubature approximations to solve difficult integrals in an approximate nonlinear Kalman filter.
- ▶ Approaches to measurement conversion with consistent covariances include using Taylor series and cubature approximations to solve difficult integrals.

Summary II

- ▶ The GNN filter is a maximum likelihood filter for data association.
- ▶ The JPDA is an MMSE (expected value) filter for data association.
- ▶ One typically uses a variant of the JPDA, because the expected value is undesirable given target identity uncertainty.
- ▶ Cascaded logic and integrated additions to GNN and JPDA filter variants allow for track initiation and termination.
- ▶ Lots of free, commented Matlab code for tracking can be found at <https://github.com/USNavalResearchLaboratory/TrackerComponentLibrary> which is also <http://www.trackercomponentlibrary.com>