

Special Functions for Optical Science and Engineering

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This book is for our special functions
Molly, Kelly, Brigid (Bridie), and Saranya.

Introduction to the Series

Since its inception in 1989, the Tutorial Texts (TT) series has grown to cover many diverse fields of science and engineering. The initial idea for the series was to make material presented in SPIE short courses available to those who could not attend and to provide a reference text for those who could. Thus, many of the texts in this series are generated by augmenting course notes with descriptive text that further illuminates the subject. In this way, the TT becomes an excellent stand-alone reference that finds a much wider audience than only short course attendees.

Tutorial Texts have grown in popularity and in the scope of material covered since 1989. They no longer necessarily stem from short courses; rather, they are often generated independently by experts in the field. They are popular because they provide a ready reference to those wishing to learn about emerging technologies or the latest information within their field. The topics within the series have grown from the initial areas of geometrical optics, optical detectors, and image processing to include the emerging fields of nanotechnology, biomedical optics, fiber optics, and laser technologies. Authors contributing to the TT series are instructed to provide introductory material so that those new to the field may use the book as a starting point to get a basic grasp of the material. It is hoped that some readers may develop sufficient interest to take a short course by the author or pursue further research in more advanced books to delve deeper into the subject.

The books in this series are distinguished from other technical monographs and textbooks in the way in which the material is presented. In keeping with the tutorial nature of the series, there is an emphasis on the use of graphical and illustrative material to better elucidate basic and advanced concepts. There is also heavy use of tabular reference data and numerous examples to further explain the concepts presented. The publishing time for the books is kept to a minimum so that the books will be as timely and up-to-date as possible. Furthermore, these introductory books are competitively priced compared to more traditional books on the same subject.

When a proposal for a text is received, each proposal is evaluated to determine the relevance of the proposed topic. This initial reviewing process has been very helpful to authors in identifying, early in the writing process, the need for additional material or other changes in approach that would serve to strengthen the text. Once a manuscript is completed, it is peer reviewed to ensure that chapters communicate accurately the essential ingredients of the science and technologies under discussion.

It is my goal to maintain the style and quality of books in the series and to further expand the topic areas to include new emerging fields as they become of interest to our reading audience.

*James A. Harrington
Rutgers University*

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Preface

To the reader:

Feed your head¹

According to a quote attributed to Albert Einstein,

“Formal symbolic representation of physical phenomena takes its rightful second place in a world where flowers and beautiful women abound.”

That being said, the language of the optical scientist/engineer is mathematical! We only symbolically represent the beauty of optical phenomena.

It also happens that we primarily deal with second-order differential equations that describe these phenomena. In many cases of interest, these second-order differential equations take certain standard forms, usually depending on the coordinate system used. These standard forms have as solutions what are known as special functions. You are familiar with elementary functions such as trigonometric functions, exponential functions, etc. These are the second stage—Bessel functions, Hermite functions, and the like. In fact, special functions have even entered the cultural zeitgeist in an episode of the popular CBS sitcom, *The Big Bang Theory*, (season 5, episode 12 “The Bus Pants Utilization,” where spherical Hankel functions are mentioned), where the main characters develop an app for smartphones that will solve differential equations in terms of special functions.

So, what do you need to get going? A basic course in calculus including, if possible, an introduction to linear differential equations (don’t worry if you are not familiar with the Frobenius method; that is described in the appendix), a familiarity with solution by separation of variables, vectors, simple trigonometry, and basic ideas of complex numbers. You will also need some elementary knowledge of optics and electrodynamics, and some knowledge of quantum mechanics will be helpful. That’s it! We have avoided complicated material such as complex variable theory, residue theorem, etc. If you are looking for rigid formalism, existence proofs, theorems, mathematical rigor and the like, you are out of luck with this book. You are better off going to

another, more sophisticated text, some of which are listed in the Bibliography. We follow Richard Feynman's advice:

“However the emphasis should be somewhat more on how to do the mathematics quickly and easily and what formulas are true, rather than the mathematician's interest in methods of rigorous proof.”

(He was commenting on operational calculus developed by Oliver Heaviside²).

Now, this is not a traditional textbook. We have tried to explain the ideas as much as we can. However, as you know by now, the only way to master physics or math is to do problems. This book does not have exercises for you to do; the main reason for this is that there are many, many books that have umpteen unsolved problems for you. We wanted to write a “readable” book so that you can get a conceptual understanding, quickly. We have also tried to give examples from a wide range of optical science and engineering. We highly recommend that you consult the references and the bibliography for more information.

We have used Python code throughout the text. Python is a public domain scripting language that is quite easy to learn and is very powerful. If you are not familiar with it, Appendix B will give you a brief introduction. We assume that you are computer literate and that you are familiar with general concepts in programming. We encourage you to run the code(s) in the chapters. All codes provided in the book stick to the ‘Minimum Working Example’ types. You are encouraged to modify these and play with them to discover for yourself the properties of these special functions. This is an integral part of this book.

We hope you enjoy this book. We certainly did enjoy working on it. We appreciate your feedback so that if there is a second edition we can incorporate your suggestions.

Finally, it is said that a book does not get finished, it escapes from the authors. That is a truism in this case. There are many aspects and applications we would have/should have included; however, there are various constraints (time, book length, energy, to name a few) that have forced us to restrict the book to its current content. To our mind, the book is good. May you, the reader, learn and enjoy!

Vasudevan Lakshminarayan
L. Srinivasa Varadharajan
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