## Contents

Preface to the Second Edition xi
Preface to the First Edition xiii
Notation for Special Functions xv

### Chapter 1. Infinite Series, Improper Integrals, and Infinite Products
1
1.1 Introduction 1
1.2 Infinite Series of Constants 2
   1.2.1 The Geometric Series 4
   1.2.2 Summary of Convergence Tests 6
   1.2.3 Operations with Series 11
   1.2.4 Factorials and Binomial Coefficients 15
1.3 Infinite Series of Functions 21
   1.3.1 Properties of Uniformly Convergent Series 23
   1.3.2 Power Series 25
   1.3.3 Sums and Products of Power Series 29
1.4 Fourier Trigonometric Series 33
   1.4.1 Cosine and Sine Series 36
1.5 Improper Integrals 39
   1.5.1 Types of Improper Integrals 39
   1.5.2 Convergence Tests 42
   1.5.3 Pointwise and Uniform Convergence 43
1.6 Asymptotic Formulas 47
   1.6.1 Small Arguments 48
   1.6.2 Large Arguments 50
1.7 Infinite Products 55
   1.7.1 Associated Infinite Series 56
   1.7.2 Products of Functions 57

### Chapter 2. The Gamma Function and Related Functions
61
2.1 Introduction 61
2.2 Gamma Function 62
   2.2.1 Integral Representations 64
   2.2.2 Legendre Duplication Formula 70
   2.2.3 Weierstrass' Infinite Product 71
2.3 Applications 77
   2.3.1 Miscellaneous Problems 77
   2.3.2 Fractional-Order Derivatives 79
2.4 Beta Function 82

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2.5 Incomplete Gamma Function 87
   2.5.1 Asymptotic Series 88

2.6 Digamma and Polygamma Functions 90
   2.6.1 Integral Representations 93
   2.6.2 Asymptotic Series 95
   2.6.3 Polygamma Functions 100
   2.6.4 Riemann Zeta Function 102

Chapter 3. Other Functions Defined by Integrals 109
3.1 Introduction 109
3.2 Error Function and Related Functions 110
   3.2.1 Asymptotic Series 112
   3.2.2 Fresnel Integrals 113
3.3 Applications 118
   3.3.1 Probability and Statistics 118
   3.3.2 Heat Conduction in Solids 119
   3.3.3 Vibrating Beams 122
3.4 Exponential Integral and Related Functions 126
   3.4.1 Logarithmic Integral 128
   3.4.2 Sine and Cosine Integrals 129
3.5 Elliptic Integrals 133
   3.5.1 Limiting Values and Series Representations 134
   3.5.2 The Pendulum Problem 135

Chapter 4. Legendre Polynomials and Related Functions 141
4.1 Introduction 141
4.2 Legendre Polynomials 142
   4.2.1 The Generating Function 142
   4.2.2 Special Values and Recurrence Formulas 146
   4.2.3 Legendre’s Differential Equation 151
4.3 Other Representations of the Legendre Polynomials 157
   4.3.1 Rodrigues’ Formula 157
   4.3.2 Laplace Integral Formula 158
   4.3.3 Some Bounds on \( P_n(x) \) 159
4.4 Legendre Series 162
   4.4.1 Orthogonality of the Polynomials 162
   4.4.2 Finite Legendre Series 165
   4.4.3 Infinite Legendre Series 167
4.5 Convergence of the Series 173
   4.5.1 Piecewise Continuous and Piecewise Smooth Functions 174
   4.5.2 Pointwise Convergence 175
4.6 Legendre Functions of the Second Kind 181
   4.6.1 Basic Properties 184
4.7 Associated Legendre Functions 186
   4.7.1 Basic Properties of \( P_n^m(x) \) 189
4.8 Applications 192
   4.8.1 Electric Potential due to a Sphere 193
   4.8.2 Steady-State Temperatures in a Sphere 197
Chapter 5. Other Orthogonal Polynomials 203

5.1 Introduction 203

5.2 Hermite Polynomials 204
  5.2.1 Recurrence Formulas 206
  5.2.2 Hermite Series 207
  5.2.3 Simple Harmonic Oscillator 209

5.3 Laguerre Polynomials 214
  5.3.1 Recurrence Formulas 215
  5.3.2 Laguerre Series 217
  5.3.3 Associated Laguerre Polynomials 218
  5.3.4 The Hydrogen Atom 221

5.4 Generalized Polynomial Sets 226
  5.4.1 Gegenbauer Polynomials 226
  5.4.2 Chebyshev Polynomials 228
  5.4.3 Jacobi Polynomials 231

Chapter 6. Bessel Functions 237

6.1 Introduction 237

6.2 Bessel Functions of the First Kind 238
  6.2.1 The Generating Function 238
  6.2.2 Bessel Functions of the Nonintegral Order 240
  6.2.3 Recurrence Formulas 242
  6.2.4 Bessel’s Differential Equation 243

6.3 Integral Representations 248
  6.3.1 Bessel’s Problem 250
  6.3.2 Geometric Problems 253

6.4 Integrals of Bessel Functions 256
  6.4.1 Indefinite Integrals 256
  6.4.2 Definite Integrals 258

6.5 Series Involving Bessel Functions 265
  6.5.1 Addition Formulas 265
  6.5.2 Orthogonality of Bessel Functions 267
  6.5.3 Fourier-Bessel Series 269

6.6 Bessel Functions of the Second Kind 273
  6.6.1 Series Expansion for $Y_n(x)$ 274
  6.6.2 Asymptotic Formulas for Small Arguments 277
  6.6.3 Recurrence Formulas 278

6.7 Differential Equations Related to Bessel’s Equation 280
  6.7.1 The Oscillating Chain 282

Chapter 7. Bessel Functions of Other Kinds 287

7.1 Introduction 287

7.2 Modified Bessel Functions 287
  7.2.1 Modified Bessel Functions of the Second Kind 290
  7.2.2 Recurrence Formulas 291
  7.2.3 Generating Function and Addition Theorems 292

7.3 Integral Relations 298
  7.3.1 Integral Representations 298
  7.3.2 Integrals of Modified Bessel Functions 299

7.4 Spherical Bessel Functions 302
  7.4.1 Recurrence Formulas 305
  7.4.2 Modified Spherical Bessel Functions 305
### Contents

#### 7.5 Other Bessel Functions
- 7.5.1 Hankel Functions 308
- 7.5.2 Struve Functions 309
- 7.5.3 Kelvin’s Functions 311
- 7.5.4 Airy Functions 312

#### 7.6 Asymptotic Formulas
- 7.6.1 Small Arguments 316
- 7.6.2 Large Arguments 317

### Chapter 8. Applications Involving Bessel Functions
- 8.1 Introduction 323
- 8.2 Problems in Mechanics
  - 8.2.1 The Lengthening Pendulum 323
  - 8.2.2 Buckling of a Long Column 327
- 8.3 Statistical Communication Theory
  - 8.3.1 Narrowband Noise and Envelope Detection 333
  - 8.3.2 Non-Rayleigh Radar Sea Clutter 336
- 8.4 Heat Conduction and Vibration Phenomena
  - 8.4.1 Radial Symmetric Problems Involving Circles 340
  - 8.4.2 Radial Symmetric Problems Involving Cylinders 343
  - 8.4.3 The Helmholtz Equation 345
- 8.5 Step-Index Optical Fibers 351

### Chapter 9. The Hypergeometric Function
- 9.1 Introduction 357
- 9.2 The Pochhammer Symbol 358
- 9.3 The Function $F(a,b;c;x)$
  - 9.3.1 Elementary Properties 362
  - 9.3.2 Integral Representation 364
  - 9.3.3 The Hypergeometric Equation 365
- 9.4 Relation to Other Functions
  - 9.4.1 Legendre Functions 373
- 9.5 Summing Series and Evaluating Integrals
  - 9.5.1 Action-Angle Variables 380

### Chapter 10. The Confluent Hypergeometric Functions
- 10.1 Introduction 385
- 10.2 The Functions $M(a;c;x)$ and $U(a;c;x)$
  - 10.2.1 Elementary Properties of $M(a;c;x)$ 386
  - 10.2.2 Confluent Hypergeometric Equation and $U(a;c;x)$ 388
  - 10.2.3 Asymptotic Formulas 390
- 10.3 Relation to Other Functions
  - 10.3.1 Hermite Functions 397
  - 10.3.2 Laguerre Functions 399
- 10.4 Whittaker Functions 403

### Chapter 11. Generalized Hypergeometric Functions
- 11.1 Introduction 411
- 11.2 The Set of Functions $\mathbf{pF}_q$
  - 11.2.1 Hypergeometric-Type Series 413
Chapter 12. Applications Involving Hypergeometric-Type Functions 429

12.1 Introduction 429

12.2 Statistical Communication Theory 429

12.2.1 Nonlinear Devices 431

12.3 Fluid Mechanics 437

12.3.1 Unsteady Hydrodynamic Flow Past an Infinite Plate 437

12.3.2 Transonic Flow and the Euler-Tricomi Equation 440

12.4 Random Fields 444

12.4.1 Structure Function of Temperature 445

Bibliography 451

Appendix: A List of Special Function Formulas 453

Selected Answers to Exercises 469

Index 473
Preface to the Second Edition

The primary changes in this second edition include the introduction of many more applications, chosen from a variety of fields such as statics, dynamics, statistical communication theory, fiber optics, heat conduction in solids, vibration phenomena, and fluid mechanics, among others. In many cases these applications appear in the chapter in which the particular special function is introduced. However, because applications involving Bessel functions and hypergeometric-type functions are far more extensive than those of the other functions, they carry over to separate chapters devoted entirely to applications (Chaps. 8 and 12).

As in the first edition, the text is suitable for use either as a classroom text in various courses dealing with higher mathematical functions or as a reference text for practicing engineers and scientists. To this end I have tried to preserve the readability of the first edition, improving it where I could by the addition of further examples or clearer exposition. For instance, I have rearranged the order of topics in Chap. 1 so that asymptotic formulas follow the discussion of improper integrals, and in addition to the chapter on applications, the discussion of Bessel functions has been expanded to two chapters—one chapter devoted entirely to Bessel functions of the first and second kinds (Chap. 6) and one devoted to Bessel functions of other kinds, such as modified Bessel functions and spherical Bessel functions (Chap. 7). These discussions on Bessel functions also include some new material such as the introduction of addition formulas, Kelvin’s functions, and Struve functions.

I am grateful to a number of students and colleagues for their helpful suggestions concerning this second edition. In particular, I wish to thank B. K. Shivamoggi, K. Vajravelu, and M. Belkerdid for their input concerning the choice of certain applications. I am further indebted to B. K. Shivamoggi for reading most of the new material.
and offering many useful suggestions. Finally, I wish to thank the entire production staff of McGraw-Hill and, in particular, acknowledge my editor, Robert Hauserman, for his continued support of this project.

L. C. Andrews
Modern engineering and physics applications demand a more thorough knowledge of applied mathematics than ever before. In particular, it is important to have a good understanding of the basic properties of special functions. These functions commonly arise in such areas of application as heat conduction, communication systems, electro-optics, nonlinear wave propagation, electromagnetic theory, quantum mechanics, approximation theory, probability theory, and electric circuit theory, among others. Special functions are sometimes discussed in certain engineering and physics courses, and mathematics courses such as partial differential equations, but the treatment of special functions in such courses is usually too brief to focus on many of the important aspects, such as the interconnecting relations between various special functions and elementary functions. This book is an attempt to present, at the elementary level, a more comprehensive treatment of special functions than can ordinarily be done within the context of another course. It provides a systematic introduction to most of the important special functions that commonly arise in practice and explores many of their salient properties. I have tried to present the special functions in a broader sense than is often done by not introducing them as simply solutions of certain differential equations. Many special functions are introduced by the generating-function method, and the governing differential equation is then obtained as one of the important properties associated with the particular function.

In addition to discussing special functions, I have injected throughout the text by way of examples and exercises some of the techniques of applied analysis that are useful in the evaluation of nonelementary integrals, summing series, and so on. All too often in practice a problem is labeled “intractable” simply because the practitioner has not been exposed to the “bag of tricks” that helps the applied analyst deal with formidable-looking mathematical expressions.

During the last 10 years or so at the University of Central Florida we have offered an introductory course in special functions to a mix of...
advanced undergraduates and first-year graduate students in mathematics, engineering, and physics. A set of lecture notes developed for that course has finally led to this textbook. The prerequisites for our course are the basic calculus sequence and a first course in differential equations. Although complex variable theory is often utilized in studying special functions, knowledge of complex variables beyond some simple algebra and Euler's formulas is not required here. By not developing special functions in the language of complex variables, the text should be accessible to a wider audience. Naturally, some of the beauty of the subject is lost by this omission.

The text is not intended to be an exhaustive treatment of special functions. It concentrates heavily on a few functions, using them as illustrative examples, rather than attempting to give equal treatment to all. For instance, an entire chapter is devoted to the Legendre polynomials (and related functions), while the other orthogonal polynomial sets, including Hermite, Laguerre, Chebyshev, Gegenbauer, and Jacobi polynomials, are all lumped together in a single separate chapter. However, once the student is familiar with Legendre polynomials (which are perhaps the simplest set) and their properties, it is easy to extend these properties to other polynomial sets. Some applications occur throughout the text, often in the exercises, and Chap. 7 is devoted entirely to applications involving boundary-value problems. Other interesting applications which lead to special functions have been omitted, since they generally presuppose knowledge beyond the stated prerequisites.

Because of the close association of infinite series and improper integrals with the special functions, a brief review of these important topics is presented in the first chapter. In addition to reviewing some familiar concepts from calculus, this first chapter contains material that is probably new to the student, such as the Cauchy product, index manipulation, asymptotic series, Fourier trigonometric series, and infinite products. Of course, our discussion of such topics is necessarily brief.

I owe a debt of gratitude to the many students who took my course on special functions over the years while this manuscript was being developed. Their patience, understanding, and helpful suggestions are greatly appreciated. I want to thank my colleague and friend, Patrick J. O’Hara, who graciously agreed on several occasions to teach from the lecture notes in their early rough form, and who made several helpful suggestions for improving the final version of the manuscript. Finally, I wish to express my appreciation to Ken Werner, Senior Editor of Scientific and Technical Books Department, for his continued faith in this project and efforts in getting it published.
### Notation for Special Functions

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<tr>
<th>Notation</th>
<th>Name of function</th>
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<td>$Ai(x)$, $Bi(x)$</td>
<td>Airy functions of the first and second kinds</td>
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<td>$bei(x)$, $ber(x)$, $bei_p(x)$, $ber_p(x)$</td>
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<td>$B(x, y)$</td>
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<td>$cn u, dn u$</td>
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<td>$D_n(x)$</td>
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<td>$EI(x), E_1(x)$</td>
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<td>$E_p(x)$</td>
<td>Weber function</td>
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<td>$G_{m, n}^{p, q}(x</td>
<td>c_q)$</td>
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</tr>
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<tr>
<td>$I_p(x)$</td>
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</tr>
<tr>
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<tr>
<td>$k_n(x)$</td>
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