Field Mathematics for Electromagnetics, Photonics, and Materials Science
A Guide for the Scientist and Engineer
**Cartesian Coordinate Expansions of Common Vector Differential Operators**

Conversions from generalized orthogonal curvilinear coordinates (GOCCs) to Cartesian:

\[ q_1 = x, \ q_2 = y, \ q_3 = z \text{ and } h_1 = 1, \ h_2 = 1, \ h_3 = 1 \]

**First-Order Vector Differential Operators (Div, Curl & Grad)**

Div vector [Eq. (4.4-22)]

\[
\nabla \cdot \mathbf{A} \bigg|_{\text{Cartesian}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{a scalar field}
\]

Div dyadic [Eq. (B.1-5)]

\[
\nabla \cdot \mathbf{G} \bigg|_{\text{Cartesian}} = \mathbf{\hat{u}}_x \left[ \frac{\partial G_{xx}}{\partial x} + \frac{\partial G_{xy}}{\partial y} + \frac{\partial G_{xz}}{\partial z} \right] + \mathbf{\hat{u}}_y \left[ \frac{\partial G_{yx}}{\partial x} + \frac{\partial G_{yy}}{\partial y} + \frac{\partial G_{yz}}{\partial z} \right] + \mathbf{\hat{u}}_z \left[ \frac{\partial G_{zx}}{\partial x} + \frac{\partial G_{zy}}{\partial y} + \frac{\partial G_{zz}}{\partial z} \right] \quad \text{a vector field}
\]

Curl vector [Eq. (4.5-12)]

\[
\nabla \times \mathbf{A} \bigg|_{\text{Cart}} = \mathbf{\hat{u}}_x \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{\hat{u}}_y \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{\hat{u}}_z \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \quad \text{a vector field}
\]

Grad scalar [Eq. (4.3-18)]

\[
\nabla V \bigg|_{\text{Cartesian}} = \mathbf{\hat{u}}_x \frac{\partial V}{\partial x} + \mathbf{\hat{u}}_y \frac{\partial V}{\partial y} + \mathbf{\hat{u}}_z \frac{\partial V}{\partial z} \quad \text{a vector field}
\]

Grad vector [Eq. (4.3-20)]

\[
\nabla \mathbf{A} \bigg|_{\text{Cartesian}} = \mathbf{\hat{u}}_{xx} \frac{\partial A_x}{\partial x} + \mathbf{\hat{u}}_{xy} \frac{\partial A_x}{\partial y} + \mathbf{\hat{u}}_{xz} \frac{\partial A_x}{\partial z} + \mathbf{\hat{u}}_{yx} \frac{\partial A_y}{\partial x} + \mathbf{\hat{u}}_{yy} \frac{\partial A_y}{\partial y} + \mathbf{\hat{u}}_{yz} \frac{\partial A_y}{\partial z} + \mathbf{\hat{u}}_{zx} \frac{\partial A_z}{\partial x} + \mathbf{\hat{u}}_{zy} \frac{\partial A_z}{\partial y} + \mathbf{\hat{u}}_{zz} \frac{\partial A_z}{\partial z} \quad \text{a dyadic field}
\]

**Second-Order Vector Differential Operators (Laplacians)**

Scalar Laplacian [Eq. (4.7-4)]

\[
\nabla^2 V \bigg|_{\text{Cartesian}} = \nabla \cdot \nabla V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad \text{a scalar field}
\]

Vector Laplacian [Eq. (4.7-11)]

\[
\nabla^2 \mathbf{A} \bigg|_{\text{Cartesian}} = \nabla \cdot \nabla \mathbf{A} = \mathbf{\hat{u}}_x \nabla^2 A_x + \mathbf{\hat{u}}_y \nabla^2 A_y + \mathbf{\hat{u}}_z \nabla^2 A_z \quad \text{a vector field}
\]

See the inside back cover for the cylindrical coordinate expansions of these operators and Appendix D for other vector differential operator expansions.
Field Mathematics for Electromagnetics, Photonics, and Materials Science
A Guide for the Scientist and Engineer

Bernard Maxum

Tutorial Texts in Optical Engineering
Volume TT64

SPIE PRESS
Bellingham, Washington USA
to my wife

Marilyn Jo
Introduction to the Series

Since its conception in 1989, the Tutorial Texts series has grown to more than 60 titles covering many diverse fields of science and engineering. When the series was started, the goal of the series was to provide a way to make the material presented in SPIE short courses available to those who could not attend, and to provide a reference text for those who could. Many of the texts in this series are generated from notes that were presented during these short courses. But as stand-alone documents, short course notes do not generally serve the student or reader well. Short course notes typically are developed on the assumption that supporting material will be presented verbally to complement the notes, which are generally written in summary form to highlight key technical topics and therefore are not intended as stand-alone documents. Additionally, the figures, tables, and other graphically formatted information accompanying the notes require the further explanation given during the instructor’s lecture. Thus, by adding the appropriate detail presented during the lecture, the course material can be read and used independently in a tutorial fashion.

What separates the books in this series from other technical monographs and textbooks is the way in which the material is presented. To keep in line with the tutorial nature of the series, many of the topics presented in these texts are followed by detailed examples that further explain the concepts presented. Many pictures and illustrations are included with each text and, where appropriate, tabular reference data are also included.

The topics within the series have grown from the initial areas of geometrical optics, optical detectors, and image processing to include the emerging fields of nanotechnology, biomedical optics, and micromachining. When a proposal for a text is received, each proposal is evaluated to determine the relevance of the proposed topic. This initial reviewing process has been very helpful to authors in identifying, early in the writing process, the need for additional material or other changes in approach that would serve to strengthen the text. Once a manuscript is completed, it is peer reviewed to ensure that chapters communicate accurately the essential ingredients of the processes and technologies under discussion.

It is my goal to maintain the style and quality of books in the series, and to further expand the topic areas to include new emerging fields as they become of interest to our reading audience.

Arthur R. Weeks, Jr.
University of Central Florida
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\]
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Bernard Maxum
October 2004
Preface

The overriding objective of this book is to offer a review of vector calculus needed for the physical sciences and engineering. This review includes necessary excursions into tensor analysis intended as the reader’s first exposure to tensors, making aspects of them understandable at the undergraduate level. A secondary objective of this book is to prepare the reader for more advanced studies in these areas.

As the world embarks on new horizons in photonics and materials science, honing one’s skills in vector calculus and learning the essential role that tensors play are paramount. New inroads in engineering are driving the need for a revamp of engineering mathematics in these areas. Profound new paradigms in optical engineering and new advances in composites are necessitating these changes. The author has found that there is an ever-increasing need for vector calculus concepts to be extended to tensors and that his undergraduates can indeed grasp tensorial concepts if taught following the lines of thinking presented here.

Whereas the classical approach to teaching electromagnetics at the junior level has been to avoid any mention of tensors, the high-tech world entering the third millennium warrants a rethinking of this practice. This is especially true as nonlinear optical effects become more common in the design of optical systems. Advanced materials, especially composites and nanodesigned materials, provide further evidence supporting the teaching of tensor fundamentals to upper-division* students. Even for isotropic materials, the fundamental relationship between stress, strain, and elastic modulus—which are rank-two and rank-four tensors—requires a fundamental understanding of tensor analysis. For anisotropic materials such as composites, piezoelectric materials, and magnetostrictive materials, tensorial relationships are unavoidable even in the linear regime.

Furthermore, the development of new photonics devices in optoelectronics, acousto-optics, magneto-optics, and fiber optics is playing an ever-increasing role in contemporary communications system design.\textsuperscript{1,2,3,4} Pollock states

\* University-level juniors and seniors.
The drive for faster systems has led to... [an] electronic speed bottleneck... This has motivated the study of integrated optics, where light, which has a much higher implicit frequency limit, is used to control light... Without a doubt the biggest research task... will be the development of optical switches and devices, and better communication architectures.

These devices include laser sources, optical switches, rare-earth-doped fiber amplifiers, nonlinear-effect fiber amplifiers, nonlinear-effect fiber soliton waves, optical detectors, and new dispersion-managed optical fibers.

**Uses of this Guide**

This is a guide, and was not planned as a text book. As such, it is intended for multiple uses, including its use as a

1. reference to salient differential and integral forms for problem solving,
2. supplement to an engineering or science course, used in conjunction with and as a counterpart to it,
3. study guide before entering such courses,
4. reference manual in an R&D laboratory or design group,
5. complement to required or elective math courses, or just as a
6. refresher and reference source to vector calculus and an introduction to tensor analysis, or a
7. text, provided the instructor devises problem sets to provide the usual practical experience with numerical examples.

**Who is this guide written for?**

Many students and working professionals experience a new awakening when they see and feel first-hand how complex mathematical concepts are applied to understanding real-world challenges. It is the intent of this guide to provide some of the mathematical prowess to facilitate reaching this level of professional elation. Other ways to state this are

*Mathematics is fun!*  
*or*  
*Knowledge is power!*
Courses such as electromagnetics (commonly called “fields”) are often viewed by students as tough and something to be avoided until late in their program. Such postponement is not advised. Other courses, such as quantum physics, fiber optic communications, nonlinear fiber optics, structural analysis, materials science or any of a number of other engineering and physical science courses are understood through exposure to the concepts of vector and tensor calculus. It is hoped that this type of exposure will provide the confidence needed to encourage students to complete mathematically intensive courses earlier in their programs by allaying their fear of an imagined mathematical abyss. In this way they will be better prepared for more advanced studies.

John R. Whinnery in his classic paper\(^{10}\) “The Teaching of Electromagnetics” states

*The set of four equations we know as “Maxwell’s equations,” in modern notation, is simple enough to imprint on a T-shirt, and yet rich enough to provide new insights throughout a lifetime of study. Some students grasp the clarity, power and excitement on first introduction while others have a very rough time with the concepts.*

Whinnery’s paper is intended to give students encouragement in approaching electromagnetics with clarity and excitement and to seek its power. His remarks might also be applied in varying degrees to other areas of physics as well, especially with regard to the mathematical constructs of Schrödinger, which are necessary for understanding the quantum physics inherent in the optical devices cited above as well as in nonlinear optical constructs.

Other challenging areas contributing to new millennium technology include

- optical communications,
- homeland security sensor systems,
- optical materials design,
- new applications of bi-anisotropic materials,
- optically based computer design for ultra-high speed and data throughput,
- space-based materials development,
- new innovations in medical imaging, and
- the design of ultra-high-bandwidth ultra-dense multi-access networks and their associated components, and
- crystal physics.
These are but few of many that could be listed.

This guide is also for anyone who is, or endeavors to be, involved in research, development, or education relating to these and other new frontiers in science and engineering. Although this guide is written with explanations and examples intended for the upper-division and first-year graduate student in science or engineering, it is also intended for those engaged in graduate research and in industrial research and development who have already been exposed to some of the concepts.

While excursions into tensors were originally written with undergraduates in mind, the author has discovered that many professionals, including academics, have a restricted understanding of tensors. A glimpse of the tensor-dyadic issue in the introduction to Chapter 3 and the rank-order issue in Section 3.1 (including the footnote), a study of Table 3-1 (at the end of Chapter 3), Table 4-1 (in Section 4.6), and especially Table 4-2 in Section 4.7 may open doors for some and provide good instructional fodder for anyone who uses tensors in their upper-division or introductory graduate courses.

**Content**

This guide consists of five chapters and four appendices. As an introduction, Chapter 1 deals with a suggested notation that distinguishes between scalars, vectors, phasors, dyadics, and higher rank tensors, without the use of boldface characters. In so doing, it briefly covers other typical notational forms that are used in this book or that one may encounter in the literature. It also covers spatial differentials and the concept, definition, and use of partial derivatives. This includes the general formulation of partial derivatives of unit vectors with respect to coordinates—a factor often neglected in undergraduate instruction leading to incorrect answers. A simple example of this concept is provided.

Chapter 2 provides a review of vector algebra covering variant and invariant scalars, scalar and vector fields, the notation and utility of phasor scalars, phasor vectors, phasor dyadics, and phasor tensors in general. Classical arithmetic vector operations of addition, subtraction, and dot-, cross-, and direct-product operations are discussed along with physical applications of these. Open and closed line and surface integrals of vector fields are cited as being potent uses of dot products in integral calculus covered in Chapter 5. Vector field direction lines and equivalence surfaces of scalar fields are also developed as further examples of the power of cross- and dot-product operations. In the process, the need for metric coefficients in coordinate expansions is introduced.
Chapter 3 gives an introduction to tensors, and the power of the use of tensor analysis is explained at a level intended for the junior, senior, or early graduate student, who may not have been previously exposed to dyadics or other tensors beyond scalars and vectors. The concept of *inner product*—a term used synonymously with dot product—is discussed. The dot products of a dyadic with a vector and a vector with a dyadic are carried out in detail, and in the process the adjective “inner” is made apparent. The dot and double-dot products of two dyadics are also detailed. These inner-product operations are expressed in their considerably more simplified tensor notation in order to illustrate the value and power of the latter.

The chapter introduces tensors of higher rank (through examples in the mechanics of materials and nonlinear optics) and the interpretation of rank in terms of “*directional compoundedness*”—a term coined by the author to help those unfamiliar with tensors to overcome the idea that a quantity can have more than one direction at every point in space and time. The rudiments of tensor analysis include rules for term-by-term rank consistency and rules for determining the resulting rank after performing certain product operations. This concept is detailed and tabulated in Chapter 4.

Chapter 4 is a review of vector calculus differential forms with excursions into tensor analysis. First-order vector differential operators are introduced with a historical perspective on the use of the “del” operator. Scalar differential operators, differential equations, and eigenvalues are generally discussed. The concepts of gradient, divergence, and curl are described in physical terms and developed from their basic definitions without the use of coordinate systems. The rank of the resultants of these first-order vector differential operators is tabulated in Table 4-1.

Vector operators of vector operators, such as the Laplacian of scalar and vector fields and six others that are commonly used in junior-level courses, are also explained in terms that conjure up images of the fields and the effects of these operators on those fields. These second-order operations are tabulated in Table 4-2 and developed in generalized orthogonal curvilinear coordinates. These are then reduced to cylindrical coordinates (rather than the usual rush to Cartesian) in order to illustrate certain terms that otherwise disappear when Cartesian coordinates are used—cylindrical being the simplest of the non-Cartesian systems, and also coincidentally being the most appropriate in the analyses of optical fibers.
Chapter 5 deals with integral forms of vector calculus and also with excursions into tensor calculus. It first delineates line and surface integrals of scalar, vector, and tensor fields with dot-, cross-, and direct-product integrands. It then covers Gauss’ divergence theorem and Stokes’ curl theorem with examples of their applications. These are first explained in physical terms and then developed mathematically. Four of the most common forms of Green’s identities are then presented, and Green’s functions are offered as a powerful approach to solving inhomogeneous partial differential equations.

Appendices

A. This appendix serves as a supplement to the vector arithmetics* covered in Section 2.4. The commutative and associative laws of vector addition and subtraction cited in Section 2.4 are demonstrated. As an application, these laws are used to show graphically and mathematically how vectors may be bisected. (Other applications of vector arithmetics can be found in Chapter 2.)

B. In this appendix divergence and curl are developed from their definitions in the more conventional Cartesian coordinates for further clarity of the concepts covered in Sections 4.4 and 4.5. The divergence is developed again in cylindrical coordinates as a first-level generalization towards curvilinear coordinates taking into account that the azimuthal \( \phi \) coordinate is the sole curvilinear coordinate in the cylindrical system. Coordinate conversions and differentials, metric coefficients, differential elements of length, and equations of coordinate surfaces are tabulated for various orthogonal coordinate systems. Finally, graphical representations of the coordinate surfaces for each specific coordinate system are displayed in perspective view following each table.

C. Intermediate-level tensor calculus is used in this appendix for the purpose of demonstrating several issues and rules cited in Chapter 3 and for providing proofs of several important postulations used in Chapter 4, especially in Tables 4.1 and 4.2. At this level we intended it for those who have learned the concepts in the earlier chapters or for those already familiar with the area. These include the proof of the Lagrange identity [Eq. (4.7-15)] that is often presented to upper-division students without such a proof. The appendix also demonstrates that the divergence operator cited by Eq. (4.7-7) is not only valid when applied to vector and dyadic

* Pronounced arith·met‘ics
operands [given by Eqs. (4.4-22) and (4.7-9), respectively] but also to any
tensor of general rank. Finally, to offset the tendency to treat the
divergence, curl, and gradient as analogous to the dot, cross, and direct
products, we emphasize that there are two properties of the nabla vector
differential operator that must both be taken into account. That is, for all
but Cartesian coordinates, the analogy is false.

D. Appendix D provides Cartesian and cylindrical coordinate expansions of
first- and second-order vector differential operators acting on scalar
(where appropriate), vector, and dyadic operands. Two applications from
materials science are presented that require the taking of the curl of the
dyadic strain and the gradient of the dyadic stress. The first yields another
dyadic, which in turn is an application of the dyadic line integral Eq.
(5.1-4). The second yields a 27-term triadic, which is explicitly provided
in Cartesian coordinates [Eq. (D.1-10)] and cylindrical coordinates [Eq.
(D.2-10)]. Several of the more common Cartesian and cylindrical
coordinate expansions presented in this appendix are listed on the inside
front and back covers of this book for the readers’ convenience.

Glossary

A glossary of the acronyms, terms, and definitions used in this book precedes the
index.

References

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