

Field Guide to

Special Functions for Engineers

Larry C. Andrews

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Field Guide to Special Functions

Field Guide to Special Functions

Most of the material chosen for this Field Guide is condensed from two textbooks: *Special Functions of Mathematics for Engineers* by L. C. Andrews and *Mathematical Techniques for Engineers and Scientists* by L. C. Andrews and R. L. Phillips. Both books are SPIE Press publications.

Many modern engineering and physics problems demand a thorough knowledge of mathematical techniques. In particular, it is important to recognize the various special functions (beyond the elementary functions) that may arise in practice as a solution to a differential equation or as a solution to some integral. It also helps to have a good understanding of the functions' basic properties. The functions treated in this Guide are among the most important for engineers and scientists. They commonly occur in problems involving electro-optics, electromagnetic theory, wave propagation, heat conduction, quantum mechanics, probability theory, and electric circuit theory, among many other areas of application.

Because of the close association of power series and improper integrals with special functions, a brief review of these important topics is included in this guide. Useful engineering functions like the step function, rectangle function, and delta (impulse) function are also introduced.

Unfortunately, notation for various engineering and special functions is not consistent among disciplines. Also, some special functions have more than one definition depending on the area of application. For these reasons, the reader is advised to be careful when using more than one reference source. The notation for the special functions adopted in this Field Guide is that which the author considers most widely used in practice.

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Glossary of Symbols and Notation

$(a)_n$	Pochhammer symbol
$\text{Ai}(x)$	Airy function of the first kind
$\text{ber}_p(x), \text{bei}_p(x)$	Kelvin's functions
$\text{Bi}(x)$	Airy function of the second kind
$B(x, y)$	Beta function
B_n	Bernoulli numbers
$B_n(x)$	Bernoulli polynomials
$\text{cn } u$	Elliptic function
$\text{comb}(x)$	Comb function
$C(x)$	Fresnel cosine integral
$\text{Ci}(x)$	Cosine integral
$C_n^\lambda(x)$	Gegenbauer polynomial
$\text{dn } u$	Elliptic function
DE	Differential equation
$\delta(x)$	Delta or impulse function
$\text{erf}(x), \text{erfc}(x)$	Error functions
$E(a_1, \dots, a_p; c_1, \dots, c_q; x)$	MacRobert E function
$E(m, \phi)$	Elliptic integral of the second kind
$\text{Ei}(x), E_1(x)$	Exponential integrals
$\mathbf{E}_p(x)$	Weber function
$F(m, \phi)$	Elliptic integral of the first kind
${}_1F_1(a; c; x)$	Confluent hypergeometric function of the first kind
${}_2F_1(a, b; c; x)$	Hypergeometric function
${}_pF_q$	Generalized hypergeometric function
$\text{Gaus}(x)$	Gaussian function
$G_{p,q}^{m,n} \left(x \begin{matrix} a_p \\ c_q \end{matrix} \right)$	Meijer G function
γ	Euler's constant
$\gamma(a, x), \Gamma(a, x)$	Incomplete gamma functions

Glossary of Symbols and Notation

$\Gamma(x)$	Gamma function
$h_p^{(1)}(x), h_p^{(2)}(x)$	Hankel spherical Bessel functions
$H_n(x)$	Hermite polynomial
$H_p^{(1)}(x), H_p^{(2)}(x)$	Hankel functions
$\mathbf{H}_p(x)$	Struve function
$i_n(x)$	Modified spherical Bessel function of the first kind
$I_p(x)$	Modified Bessel function of the first kind
$j_n(x)$	Spherical Bessel function of the first kind
$\mathbf{Ji}_p(x)$	Integral Bessel function
$J_p(x)$	Bessel function of the first kind
$\mathbf{J}_p(x)$	Anger function
$\ker_p(x), \text{kei}_p(x)$	Kelvin's functions
$k_n(x)$	Modified spherical Bessel function of the second kind
$K_p(x)$	Modified Bessel function of the second kind
$\text{li}(x)$	Logarithmic integral
$L_n(x)$	Laguerre polynomial
$L_n^{(a)}(x)$	Associated Laguerre polynomial
$\mathbf{L}_p(x)$	Modified Struve function
$\Lambda(x)$	Triangle function
$M_{k,m}(x)$	Whittaker function of the first kind
$n!$	Factorial function
$P_n(x)$	Legendre polynomial
$P_n^m(x)$	Associated Legendre function of the first kind
$P_n^{(a,b)}(x)$	Jacobi polynomial
$\Pi(x)$	Rectangle function
$\Pi(m, \phi, a)$	Elliptic integral of the third kind
$Q_n(x)$	Legendre function of the second kind
$Q_n^m(x)$	Associated Legendre function of the second kind
$\text{rect}(x)$	Rectangle function
$S(x)$	Fresnel sine integral

Glossary of Symbols and Notation

$\text{Si}(x), \text{si}(x)$	Sine integral
$\text{sgn}(x)$	Signum (sign) function
$\text{sinc}(x)$	Sinc function
$\text{step}(x)$	Step function
$\text{tri}(x)$	Triangle function
$T_n(x)$	Chebyshev polynomial of the first kind
$U(x)$	Step function
$U(a; c; x)$	Confluent hypergeometric function of the second kind
$U_n(x)$	Chebyshev polynomial of the second kind
$W_{k,m}(x)$	Whittaker function of the second kind
$y_n(x)$	Spherical Bessel function of the second kind
$Y_p(x)$	Bessel function of the second kind
$Y_n^m(\theta, \phi)$	Spherical harmonic
$\psi(x)$	Digamma (psi) function
$\psi^{(m)}(x)$	Polygamma function
$\zeta(x)$	Riemann zeta function
\sim	“... is asymptotic to”
$ $	Absolute value
$\binom{a}{n}$	Binomial coefficient