Generalization of the first-order formula for analysis of scan patterns of Risley prisms

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Abstract. A first-order formula for calculations of the direction cosines of the rays refracted by Risley prisms was derived. The formula was obtained by representing the deviation of the ray passing through a prism by the product of rotation matrices, and using the series expansion of the product. It can be applied to the system of an arbitrary number of prisms or a combination of Risley prisms. Related errors were discussed and some numerical calculations were made and compared with the exact solutions using the refraction equation. The scan patterns of a single Risley prism or a combination of two Risley prisms calculated using the generalized first-order formula are in good agreement with the exact solutions.

Subject terms: Risley prism; deviation; beam scanning; scan pattern; first-order approximation.

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1 Introduction

Risley prisms, composed of two prisms with a small apex angle, are widely used for beam scanning or steering in optical instruments and other developing systems. The beam steering using three prisms is also investigated for the same applications. The primary concern for analyses of Risley prisms is to calculate the deviations of the ray passing through them, thereby obtaining the steering or scan patterns. The basic properties of Risley prisms can be understood as the combination of deviations by each prism, but the rigorous calculation is complicated because several refractions at planar surfaces are involved. In previous works, several methods have been used such as the three-dimensional model and analytic formulas and approximate formulas. Recently, approximate formulas up to third-order of the apex angle were obtained by expanding analytic solution. But all the analytic formulas and the approximate ones in those works were obtained for a single Risley prism composed of two prisms. In this paper, by representing the deviation of a ray passing through a prism by the product of rotation matrices, a generalized first-order formula was obtained. It can be applied to the system of an arbitrary number of prisms or combination of Risley prisms. Related errors were discussed and some numerical calculations were made and compared with the exact solutions using the refraction equation. The scan patterns of a single Risley prism or a combination of two Risley prisms were calculated using the generalized first-order formula, and the results were in good agreement with the exact solutions.

2 Refraction Equation for Cascaded Planar Surfaces

Let \( s_i \) and \( s'_i \) be unit vectors in the direction of incident and transmitted rays at \( i \)th surface, and \( N_i = (\sin \alpha_i, \cos \phi_i, \sin \alpha_i, \sin \phi_i, \cos \alpha_i) \) is the unit normal vector at this surface (see Fig. 1), then Snell’s law can be written as:

\[
n_i N_i \times s_i = n'_i N_i \times s'_i.
\]

where \( n_i \) and \( n'_i \) are refractive indices of the mediums before and after the \( i \)th surface. Performing vector product with \( N_i \) on both sides of Eq. (1) and using the vector identity:

\[
A \times (B \times C) = (A \cdot C)B - (A \cdot B)C,
\]

we obtain

\[
n_i [(N_i \cdot s_i) N_i - s_i] = n'_i [(N_i \cdot s'_i) N_i - s'_i].
\]

This gives

\[
s'_i = \gamma'_i N_i - \frac{n_i}{n'_i} (\gamma_i N_i - s_i),
\]

where the following definitions are used:

\[
\gamma_i \equiv N_i \cdot s_i = \cos \Theta_i, \quad \gamma'_i \equiv N_i \cdot s'_i = \cos \Theta'_i,
\]

where \( \Theta_i \) is the angle between \( N_i \) and \( s_i \) and \( \Theta'_i \) is defined similarly. From Snell’s law: \( n'_i \sin \Theta'_i = n_i \sin \Theta_i \) and Eq. (4) we have

\[
\gamma'_i = \sqrt{1 - \sin^2 \Theta'_i} = \frac{1}{n'_i} \sqrt{n_i^2 - n_i^2 \gamma_i^2}.
\]

Equation (3) with coefficients given by Eq. (5) is the refraction equation at \( i \)th surface. For successive calculations, we put \( \gamma_{i+1} = \gamma_i \) and \( n_{i+1} = n_i \). By applying the refraction equation successively to each surface, we can obtain directions of a ray passing through an arbitrary number of surfaces. We notice that the components of the ray vector \( s_i = (x_{s_i}, y_{s_i}, z_{s_i}) \) are direction cosines of the ray. The components of the refracted ray vector are represented as \( s'_i = (x'_{s_i}, y'_{s_i}, z'_{s_i}) \). Normal vectors and ray vectors for analysis of a typical Risley prism are shown in Fig. 2. We define the rotation angle \( \phi_1 \) of each prism to be zero when the apex is directed downward along the \( x \) axis as shown in Fig. 2. When the second and third surfaces are orthogonal to the \( z \) axis, the azimuth angles of normal vectors \( N_1 \) and \( N_2 \) are related to the rotation angles by

\[
\phi_1 = \varphi_1, \quad \phi_4 = \pi + \varphi_2.
\]
For the configuration of Fig. 2 with \( \phi_2 = \pi \), we have \( \phi_4 = 2\pi \), which means \( N_4 = N_1 \) when the apex angles are same, as expected. In Sec. 3 we will use Eqs. (3)-(5) for numerical calculations.

Substituting Eq. (5) into Eq. (3) gives

\[
s'_1 = ns_1 + \sqrt{1 - n^2 + n^2 \gamma_1^2} N_1 - n \gamma_1 N_1, \quad (n) = n / n'. \]

The refraction equation in this form is used for analysis of the Risley prisms \(^n\) and also for analyses of the plano convex or hyperboloidal focusing lenses \(^o\).

### 3 Derivation of the Approximate Formula

By applying Eqs. (3)-(5) to two successive surfaces of a prism with refractive index \( n \), we obtain

\[
s'_2 = (n' - n)(n' \gamma_1 - \gamma_1) N_2 + (n' \gamma_1 - \gamma_1) N_1 + s_1, \quad (7)
\]

which means the vector \( s'_2 - s_1 \) is in the plane generated by \( N_1 \) and \( N_2 \). So, if \( N_1 \) and \( N_2 \) are in the \( x-z \) plane, then the effect of the prism is the rotation of the ray vector \( s_1 \) about the \( y \)-axis, and it is clear that the rotation is counter-clockwise when the deviation angle \( \phi \) is zero. Let \( \delta \) be the deviation angle of the prism, then this rotation can be represented in matrix form as follows:

\[
R_{s}(\delta) = \begin{pmatrix} \cos \delta & 0 & -\sin \delta \\ 0 & 1 & 0 \\ -\sin \delta & 0 & \cos \delta \end{pmatrix}. \quad (8)
\]

To find the transmitted ray vector for a prism rotated by \( \phi \) in azimuth, the components of the incident ray vector have to be transformed to the coordinate system with \( \phi = 0 \), i.e., the coordinate system fixed to the prism, then rotated about the \( y \) axis by \( \delta \) and transformed to the original coordinate system. This operation can be represented as follows:

\[
M_s(\phi) = R_{s}(\phi) R_{y}(\delta) R_{z}(-\phi). \quad (9)
\]

Here the matrix \( R_{s}(\phi) \) for the rotation about the \( z \) axis is

\[
R_{z}(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (10)
\]

The incident ray vector \( s_1 = (s_{1x}, s_{1y}, s_{1z}) \) and the transmitted ray vector \( s = (s_x, s_y, s_z) \) which is equal to \( s'_2 \) of Eq. (8) are related by

\[
s = M_s(\phi) s'_1, \quad (11)
\]

where the superscript \( T \) represents the transpose operation for the components of vector. Substituting Eq. (5) into Eq. (9) and using the series expansion about \( \delta \) gives

\[
M_s(\phi) = M_s^{(1)}(\phi) - \frac{1}{2} \delta^2 M_s^{(2)}(\phi) + O(\delta^3), \quad (12)
\]

where the matrices are defined by

\[
M_s^{(1)}(\phi) = \begin{pmatrix} 1 & 0 & \delta \cos \phi \\ 0 & 1 & -\delta \sin \phi \\ 0 & 0 & 1 \end{pmatrix}, \quad (13)
\]

and

\[
M_s^{(2)}(\phi) = \begin{pmatrix} \cos^2 \phi & \cos \phi \sin \phi & 0 \\ \cos \phi \sin \phi & \sin^2 \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (14)
\]

and \( I \) in Eq. (12) is the unit matrix.

When the incident ray is in the \( x-z \) plane, i.e., \( s_{1y} = 0 \), the deviation angle \( \delta \) is given by

\[
\delta = \Theta - \alpha + \sin^{-1}[\sin \alpha \cdot (n^2 - \sin^2 \Theta)^{1/2} - \sin \Theta \cos \alpha]. \quad (15)
\]

When \( s_{1y} \) is not zero, the deviation angle must be defined to be the angle between the vectors projected on the \( x-z \) plane, i.e., \( s_{ip} \equiv (s_{1x}, 0, s_{1z}) \) and \( s_{p} \equiv (s_{x}, 0, s_{z}) \). If the angle between \( s_{ip} \) and \( N_1 \) is denoted by \( \Theta_p \), and the angle between \( s_{ip} \) and \( s_1 \) by \( \Theta \), as shown in Fig. 3, then the corresponding angles \( \Theta_p \) and \( \Theta' \) for the refracted ray \( s'_1 \) are given by Snell’s law in the following form:

\[
n_r \sin \Theta_p \cos \Theta \equiv n'_r \sin \Theta'_p \cos \Theta' \quad (16a)
\]

\[
n_r \sin \Theta_v = n'_r \sin \Theta' \quad (16b)
\]

where \( n_r \) and \( n'_r \) are refractive indices of the mediums before and after the surface. By using Eq. (16a) with \( n_1 = 1 \) and \( n'_1 = n \) and following the same method for derivation of Eq. (15) we obtain

\[
\delta = \Theta_p - \alpha + \sin^{-1}[\sin \alpha \cdot (n^2 - \psi^2)^{1/2} - \sin \Theta_p \cos \alpha], \quad (17)
\]

where \( \psi = \cos \Theta'/\cos \Theta \). Using Eq. (16b) we have \( \psi \approx 1 + [(n^2 - 1)/(2n^2)] \Theta_p^2 \). When \( \Theta_v \) is zero, we have \( \Theta_p \)
Jeon: Generalization of the first-order formula for analysis of scan patterns...

By expanding Eq. (17) up to the third-order terms, we obtain
\[
\delta = (n - 1)\alpha + (v/n)\alpha \Theta_p^2 - v\alpha^2 \Theta_p
+ (v/n)\alpha \Theta_p^2 + (n/3)\alpha^3, \tag{18}
\]
where the constant \( v \) is defined by \( v = (1/2)(\nu^2 - 1) \). The first-order approximation of \( \delta \) is \( \delta(1) = (n - 1)\alpha \) as known, and Eq. (18) can be written as
\[
\delta = \delta(1) + \epsilon(3), \tag{19}
\]
where \( \epsilon(3) \) is the third-order term:
\[
\epsilon(3) = (v/n)\alpha \Theta_p^2 - v\alpha^2 \Theta_p + (v/n)\alpha \Theta_p^2 + (n/3)\alpha^3. \tag{20}
\]

Using \( \delta = \delta(1) + \epsilon(3) \), and \( s_{1z} \approx \sin(\alpha) \) and \( s_{1z} \approx 1 \) in the higher order terms, Eq. (20) becomes
\[
s_x = s_{1x} + \delta \cos \phi \cdot s_{1z} - (1/2)\delta^2 (s_{1x} \cos^2 \beta + s_{1y} \cos \beta \sin \beta) + O(\delta^3 s_{1x}),

s_y = s_{1y} + \delta \sin \phi \cdot s_{1z} - (1/2)\delta^2 (s_{1x} \cos \beta \sin \beta + s_{1y} \sin^2 \beta) + O(\delta^3 s_{1y}),

s_z = s_{1z} - \delta \cos \phi \cdot s_{1x} - \delta \sin \phi \cdot s_{1y} - (1/2)\delta^2 s_{1z} + O(\delta^3 s_{1z}). \tag{21}
\]

Equation (21) means that if we use the first-order approximation \( M_3(\phi) \), i.e.,
\[
M_3(\phi) = \begin{pmatrix} 1 & 0 & \delta(1) \cos \phi \\ 0 & 1 & \delta(1) \sin \phi \\ -\delta(1) \cos \phi & -\delta(1) \sin \phi & 1 \end{pmatrix}, \tag{22}
\]
then the errors in calculating \( s_x \) and \( s_y \) are determined by the third-order terms of \( \alpha \) and \( \Theta \), and the one for \( s_z \) is determined by the second-order term of \( \alpha \). The last term in the equation of \( s_x \) or \( s_y \) in Eq. (21) depends on \( \Theta \), so that the errors do not vanish even in the case of \( \epsilon(3) = 0 \). For example, when \( n = 1.5, \alpha = 0.2 \) rad, and \( \Theta = 0.1 \) rad, it is \((n - 1)^2 \alpha^2 \Theta = 1.0 \) mrad. The order of magnitude of the third terms including \( \epsilon(3) \) are estimated by assuming \( \phi = 0 \) rad and \( s_{1z} = 1 \). Figure 4 shows that \( \epsilon(3) \approx 1.7 \) mrad when \( \Theta_p = 0.1 \) rad, from which the total error in calculation of \( s_x \) or \( s_y \) using the first-order formula is estimated to be 2.7 mrad. When several prisms are involved, the total error depends on the relative orientations of the prisms, and the error analysis done here will give only the order of magnitudes.

Hereafter, we will use Eq. (22) for calculations of ray vectors, and also use \( \delta = \delta(1) \) by dropping the upper index. To investigate the scan patterns, only \( s_x \) and \( s_y \) are needed, so that the errors of our first-order formula are of third-order.

For analysis of a Risley prism, let the deviation angle of the first prism be \( \delta_1 \), and the one of the second prism be \( \delta_2 \), and the rotation angle of each prism be \( \phi_1 \) and \( \phi_2 \). Since \( M_3(\phi) \) in Eq. (22) is independent of incident ray vectors, the transmitted ray vector \( s \) can be obtained from the following equation:
\[
s^T = M_3(\phi_2)M_3(\phi_1)s_{1p}^T \equiv \tilde{M}s^T. \tag{23}
\]

Substituting Eq. (22) into (23) and keeping the terms of the first-order with respect to \( \delta_1 \) or \( \delta_2 \), we have
\[ M = \begin{pmatrix} 1 & 0 & \delta_1 \cos \varphi_1 + \delta_2 \cos \varphi_2 \\ 0 & 1 & \delta_1 \sin \varphi_1 + \delta_2 \sin \varphi_2 \\ -\delta_1 \cos \varphi_1 - \delta_2 \cos \varphi_2 & -\delta_1 \sin \varphi_1 - \delta_2 \sin \varphi_2 & 1 \end{pmatrix}. \tag{24} \]

It is straightforward to generalize Eq. (25) to the formula for a system composed of arbitrary number of prisms, and the result is

\[ s_x = s_{1x} + \left( \sum_{i=1}^{N_1} \delta_i \cos \varphi_i \right) s_{2x}, \]
\[ s_y = s_{1y} + \left( \sum_{i=1}^{N_1} \delta_i \sin \varphi_i \right) s_{2y}, \tag{30} \]
\[ s_z = -\left( \sum_{i=1}^{N_1} \delta_i \sin \varphi_i \right) s_{2x} + s_{2z}. \]

where \( N \) is the number of prisms, and \( \delta_i \) is the deviation angle, and \( \varphi_i \) is the rotation angle for the \( i \)th prism.

### 4 Numerical Calculations for Sample Cases

To obtain the scan patterns of a Risley prism, we put the rotational frequencies of the prisms to be \( f_1 \) and \( f_2 \) each so that the angles of rotations are

\[ \psi_1(t) = \psi^{(i)}_1 + 2\pi f_1 t, \quad \psi_2(t) = \psi^{(i)}_2 + 2\pi f_2 t, \tag{31} \]

with time \( t \) and initial angles \( \psi^{(i)}_1 \) and \( \psi^{(i)}_2 \). The scan patterns depend on the initial angles of rotations and the following ratios:

\[ k = \frac{\delta_2}{\delta_1}, \quad M = \frac{f_2}{f_1}. \tag{32} \]

For the first case, we consider the configuration where the second prism rotates in the opposite direction with the same frequency, i.e., \( f_2 = -f_1 \), and the initial angles are \( \psi^{(i)}_1 = 0 \) and \( \psi^{(i)}_2 = \pi \) so that we have \( \psi_2(t) = \pi - \psi_1(t) \). Let the deviation angles be the same; \( \delta_2 = \delta_1 \). This case corresponds to \( k = 1 \) and \( M = -1 \). From Eq. (26) we obtain the approximate solutions for the components of the transmitted ray vector as follows:

\[ s_x = 0, \quad s_y = 2\delta_1 \sin(2\pi f_1 t), \quad s_z = 1, \tag{33} \]

which means the linear scan along the \( y \) axis. In this paper, all the refractive indices are assumed to be \( n = 1.5 \), and the apex angle of prism 1 to be \( \alpha_1 = 0.2 \) rad, so that \( \delta_1 = (n-1)\alpha_1 = 0.1 \) rad. The exact numerical calculation using Eqs. [3] [5] can be performed straightforwardly, where the azimuth angles of normal vectors are determined by Eq. [6].

Figure 5 shows the scan patterns generated during one period of rotation for prism 1. The maximum error in the \( x \) direction is about 1.6 mrad at \( s_y \approx 0.17 \), which is comparable with the total error (2.7 mrad) obtained for a single prism in...
Sec. 4 The bow tie pattern of the exact solution is one of the typical properties of the Risley prisms with the same apex angle $\theta$.

Figure 5 is an example of scan patterns obtained by the approximate calculations using Eqs. (23) and the exact numerical calculations using Eqs. (24)–(25), in which we consider the axial and the oblique incident rays which are specified by $s_i = (\sin \theta, 0, \cos \theta)$ with the polar angles $\theta = 0.0$ and 0.1 rad each. The case of $\theta = 0.0$ rad is comparable to the one given in Ref. 11. It can be seen that the error of the approximate solution for $\theta = 0.0$ rad at $s_i = 0.0$ and $s_r = 0.17$ is approximately 2.8 mrad in the $y$ direction, and the one for $\theta = 0.1$ rad is approximately 4.2 mrad in the $x$ direction. The errors in the $x$ direction are approximately zero in both cases. The approximate solutions are in reasonably good agreement with the exact solutions even though the errors increase with the polar angle $\theta$.

When two Risley prisms are combined as in Fig. 1, it can generate more general two-dimensional scan patterns. As an example, we use the configuration in which the second Risley prism is rotated by 90 deg, and the apex angles of prisms in each Risley prism are the same. Here the directions of rotation in each Risley prism are opposite. Let the deviation angles of the prisms in each Risley prism be $\delta_A$ and $\delta_B$, and the rotation angle be $\varphi_A$ and $\varphi_B$. Therefore we put $\delta_1 = \delta_2 = \delta_A$, $\varphi_1 = \varphi_A$, $\varphi_2 = \pi - \varphi_A$ for Risley prism A. Using Eq. (34) we obtain

$$\tilde{M}_A(\varphi_A) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2\delta_A \sin \varphi_A \\
0 & -2\delta_A \sin \varphi_A & 1
\end{pmatrix}. \quad (34)$$

For Risley prism B, we put $\delta_1 = \delta_2 = \delta_B$, $\varphi_1 = (\pi/2) + \varphi_B$, $\varphi_2 = -(\pi/2) - \varphi_B$, so that we obtain

$$\tilde{M}_B(\varphi_B) = \begin{pmatrix}
1 & 0 & -2\delta_B \cos \varphi_B \\
0 & 1 & 0 \\
2\delta_B \cos \varphi_B & 0 & 1
\end{pmatrix}. \quad (35)$$

The transmitted ray vector $s = (s_1, s_2, s_3)$, when the incident ray vector is $s_i$, is given by the following equation:

$$s^T = M_b(\varphi_B) M_a(\varphi_A) s_i^T. \quad (36)$$

When the incident ray is axial, substituting Eqs. (34) and (35) into Eq. (36) gives

$$s_1 = -2\delta_B \sin \varphi_B, \quad s_2 = 2\delta_A \sin \varphi_A, \quad s_3 = 1. \quad (37)$$

To obtain a scan pattern, we put the rotational frequencies to be $f_A$ and $f_B$ for each Risley prism so that

$$\varphi_A(t) = 2\pi f_A t, \quad \varphi_B(t) = 2\pi f_B t. \quad (38)$$

The ratios similar to the ones in Eq. (32) can be defined by

$$k = \delta_B/\delta_A, \quad M = f_B/f_A. \quad (39)$$

For exact calculation using the refraction equation, we need to specify the rotation angles of each prism. Let $\varphi_{A1}$ and $\varphi_{A2}$ be rotation angles of prisms in Risley prism A (referring to Fig. 7), and $\varphi_{B1}$ and $\varphi_{B2}$ be the ones of prisms in Risley prism B, then we have

$$\varphi_{A1}(t) = 2\pi f_A t, \quad \varphi_{A2}(t) = \pi - 2\pi f_A t, \quad (40)$$

Using Eqs. (30) and (40) we can determine all the components of the normal vectors, and perform exact calculations using Eqs. (33)–(35).
Figure 8 shows the scan patterns generated during one period of rotation of the prism A1 for the two cases of $M = 6$, $k = 1$, and $M = 7$, $k = 1$.

It can be seen that the errors of the approximate solutions for $M = 6$ are about 10 mrad in the $x$ direction and 7 mrad in the $y$ direction at the point of $s_x = 0.2$ and $s_y = 0.2$. The errors of the same level of magnitudes are obtained for the case of $M = 7$. We can notice that the approximate solutions have reasonable accuracies for describing the scan patterns. It is also observed that the scan patterns obtained from this configuration are composed of closed curves when $M$ is even.

5 Conclusion

A first-order formula for calculations of the direction cosines of the rays refracted by Risley prisms was derived. The
Jeon: Generalization of the first-order formula for analysis of scan patterns. The first-order formula was obtained by representing the deviation of the ray passing through a prism by the product of rotation matrices, and using the series expansion of the product. It can be applied to the system of arbitrary number of prisms or combination of Risley prisms. It permits the calculations of the direction cosines of the transmitted ray vectors for arbitrary incident rays such as oblique rays. The errors associated with the first-order formula were analyzed by using the series expansion of the expression for the deviation angle. It showed that the errors are of third-order of the prism’s apex angle and the incidence angle. The numerical estimation using examples showed that the total error for a single prism is reasonably small, approximately 2.7 mrad for incident angles of 0.2 rad. The generalized first-order formula was applied to the numerical calculations of the scan patterns of a single Risley prism and a combination of two Risley prisms, and the results were compared with the exact solutions using the formulation based on the refraction equations. The maximum error in the scan patterns of the single Risley prism in the example was 2.8 mrad for axial incident rays and 4.2 mrad for oblique rays with the polar angle of 0.1 rad. The maximum error in the scan patterns of the combination of two Risley prisms in the example were about 10 mrad in the x direction at the point of $s_x = 0.2$ and $s_y = 0.2$. Even though the errors tend to increase with the number of prisms, the first-order approximate formula will be useful for analyzing the scan patterns of Risley prisms.

References


Yong-Geun Jeon received his PhD degree in physics from Korea Advanced Institute of Science and Technology in 1994. The topic of his PhD research was the stimulated Raman and Brillouin scattering in high pressure gases. He is now a principal researcher of the Agency for Defense Development. He has been working on the developments of electro-optical systems. His main research interests are solid-state lasers, nonlinear optics, and the optical design for laser systems. He is a member of the OSA.