Forum

Infrared

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Efficient computation of blackbody functions

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Johnson and Branstetter[1] favor Laguerre-Gauss quadrature over the usual series summation[2] for the fast, precise evaluation of integrals of the Planck function. But the only disadvantage of the usual series, which is an expansion in powers of exp (-x), is that is does not converge very rapidly for small x. This problem is easily solved through the use, for small x, of a more rapidly converging expansion in powers of x. The two series together provide more precision than the Laguerre-Gauss quadrature for a comparable number of terms, and are fast and simple to use. This note summarizes the expressions needed to obtain any required accuracy with the two-series approach. Expressions are given which bound the fractional residual errors when the series are truncated; and to show the precision attainable, these bounds are plotted as functions of x for 5 and 10 terms of each series.

The radiant energy emitted by a blackbody of unit area, in the wavelength interval from 0 to λ, is, using Makowski's notation,[2]

$$H_{o-\lambda} \equiv \int_o^\lambda H_\lambda \, d\lambda = \int_o^\lambda \frac{c_1 \, d\lambda}{\lambda^5 \left[e^{c_2/\lambda T} - 1 \right]} \tag{1}$$

where $c_1 = 2\pi hc^2$ and $c_2 = hc/k$ are the first and second radiation constants. The total energy emitted at all wavelengths is

$$H \equiv H_{o-\infty} = \sigma T^4 \tag{2}$$

where $\sigma = \pi^4 c_1 / 15\, c_2^4$ is the Stefan-Boltzmann constant.

The series commonly used to compute this function is obtained by putting $x = c_2/\lambda T$, to yield

$$\frac{H_{o-\lambda}}{H} = \frac{15}{\pi^4} \int_x^\infty \frac{x^3 \, dx}{e^x - 1} \, . \tag{3}$$

The denominator of the integrand is then expanded in powers of exp (-x) with the binomial theorem, and integrated by parts. The result is

$$\frac{H_{o-\lambda}}{H} = \frac{15}{\pi^4} \sum_{n=1}^{\infty} \frac{e^{-nx}}{n^4} \left[(nx)^3 + 3(nx)^2 + 6(nx) + 6 \right] \, . \tag{4}$$

This series converges for all $x \geq 0$, and thus for all possible values of wavelength and temperature. A bound for the error r_N made by truncating the series after N terms is easily obtained. If the general term of the series is defined by g (nx), then

$$r_N = \frac{15}{\pi^4} \sum_{n=N+1}^{\infty} g\,(nx) \, . \tag{5}$$

The exponential factor dominates the behavior of g (nx); hence the polynomial factor may be replaced in each term by a larger constant; thus:

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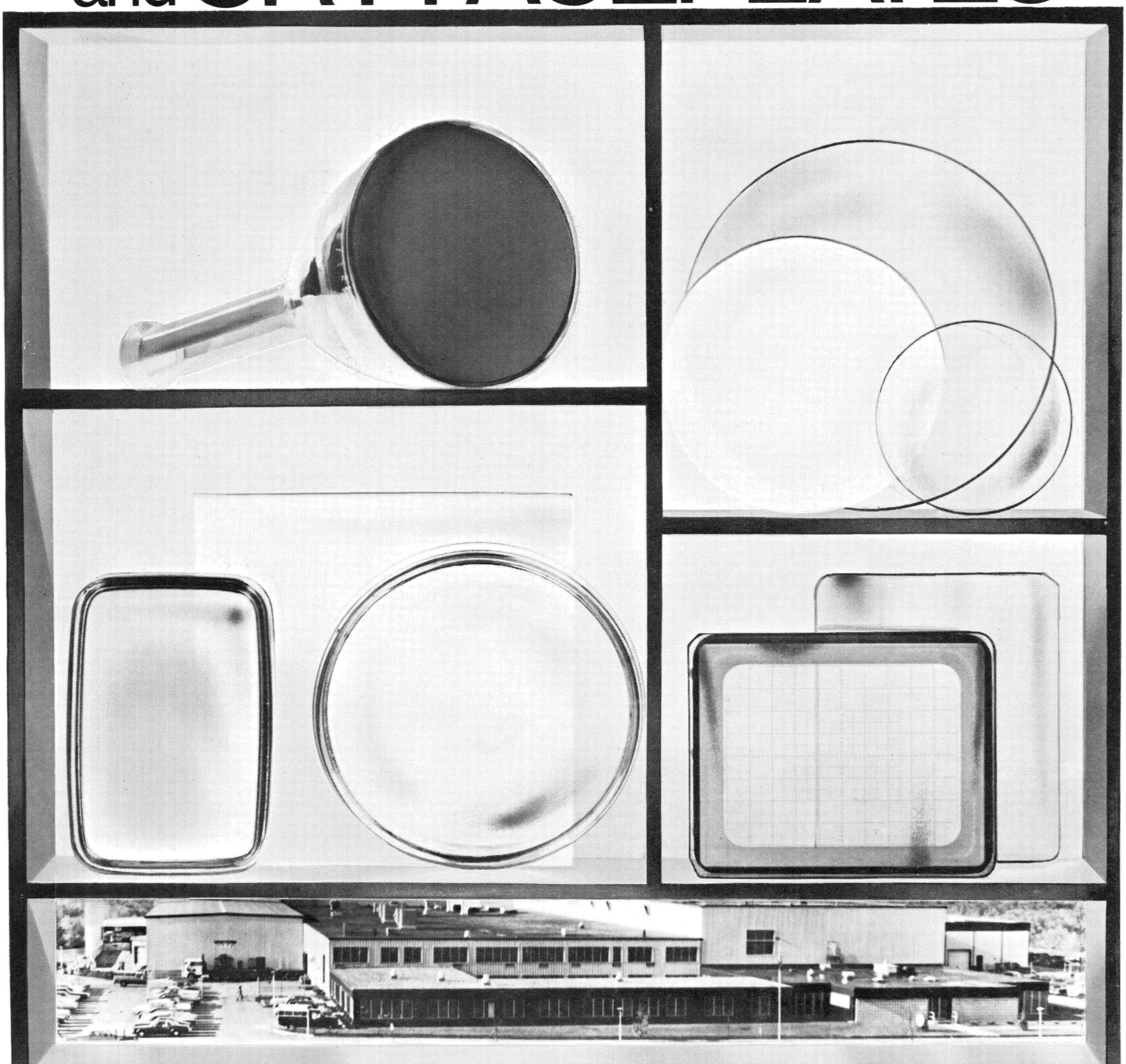

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$$r_N < \frac{15}{\pi^4}\left[\frac{x^3}{(N+1)}+\frac{3x^2}{(N+1)^2}+\frac{6x}{(N+1)^3}+\frac{6}{(N+1)^4}\right]\sum_{n=N+1}^{\infty} e^{-nx} \,. \tag{6}$$

The remaining series is geometric and may be written in closed form. The remainder may thus be written:

$$r_N < \frac{15}{\pi^4}\;\frac{g\,[(N+1)\,x]}{1-e^{-x}} \,. \tag{7}$$

The fractional error is:

$$\epsilon_N < \frac{r_N}{(H_{o\text{-}\lambda}/H)} < \frac{g\,[(N+1)\,x]}{\sum\limits_{n=1}^{N} g\,(nx)\,(1-e^{-x})} \,. \tag{8}$$

The integrated radiant emittance, $H_{O\,-\,\lambda}$, may be obtained from the series of Eq. (4) to any precision desired by programming the last inequality of Eq. (8) as an automatic test. The series is convergent for all $x \geq 0$, and the test fails only when $x = 0$. The only problem with this approach is that for small x, convergence is very slow.

Fortunately, an expansion in powers of x, which converges quickly for small x, is available. The Bernoulli numbers are sometimes defined by the equation[3]:

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}B_{2n-1}}{(2n)!}\,x^{2n} \,. \tag{9}$$

This series converges for all $x < 2\pi$. In the notation used, the nonzero Bernoulli numbers are all positive and odd numbered. The first few are $B_1 = 1/6$, $B_3 = 1/30$, $B_5 = 1/42$.

This series permits efficient computation of the radiant emittance for small x. From Eqs. (2) and (3),

$$\frac{H_{o\text{-}\lambda}}{H} = 1 - \frac{15}{\pi^4}\int_o^x \frac{x^3\,dx}{(e^x-1)} \,. \tag{10}$$

Substituting the power series from Eq. (9) into Eq. (10) and integrating,

$$\frac{H_{o\text{-}\lambda}}{H} = 1 - \frac{15}{\pi^4}x^3\left[\frac{1}{3}-\frac{x}{8}+\sum_{n=1}^{\infty}\frac{(1)^{n+1}B_{2n-1}}{(2n+3)(2n)!}\,x^{2n}\right] \,. \tag{11}$$

Since this series alternates, the truncation error is smaller in magnitude than the first neglected term. Thus,

$$r_N < \frac{15}{\pi^4}\;\frac{B_{2N+1}\,x^{2N+5}}{(2N+5)(2N+2)!} \tag{12}$$

and the fractional error is, as before,

$$\epsilon_N = \frac{r_N}{(H_{o\text{-}\lambda}/H)} \,. \tag{13}$$

The bounds for the fractional errors made by truncating the two series after 5 and 10 terms are shown in Figure 1 as functions of x. The fractional errors for the small-x series decrease with x, and those for the large-x series decrease with 1/x. Thus, by using both series, with an appropriate division point, precise results can be obtained with reasonable effort. For example, if the computation is split at $x = 1.75$, and five terms of each series are used, the fractional error will not exceed 10^{-5}. If the computation is split at $x = 1.95$ and ten terms of each series are used, the fractional error will not exceed 10^{-10}. In both cases the precision exceeds that obtainable with Laguerre-Gauss quadrature using the same number of terms.[1]

Use of the series in Eq. (11) to compute $H_{o\text{-}\lambda}$ to some prescribed precision, without regard to the number of terms required, can be awkward. Such a procedure will necessitate the storage of many Bernoulli numbers, or the inclusion of a series to compute them. It may be more convenient, particularly with hand-held calculators, to use a fixed number of terms of this series. Thus note from Figure 1 that if the computation is split at $x = 1.2$, five terms of the small-x series will ensure a fractional error smaller than 10^{-7}, and five terms are easily programmed. If $x > 1.2$, the large-x series can be programmed, with an automatic test, to provide comparable precision using at most 10 terms. This combination thus provides quite good precision with computational simplicity.

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The History of Optics

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The greatest authority on optics in the Middle Ages

Our comprehension of scientific achievements is often limited to those developments recorded in one's own country or in those allied to it. That this is so is probably attributable to patriotic chauvinism and religious prejudice. For instance, we are likely to consider that the optics we practice (as well as many other cultural activities) began in Greece and came to us directly by way of Europe. Let us consider the Arabian influence on optics and dispel some of this myth.

About A.D. 575 in the Arabic city of Mecca a humble boy was born, who came to be known as Mohammed (which is roughly translated as *Highly Praised*). He founded the Muslim faith and revitalized the pursuit of culture in the Arabic countries. In this way, the Arabs became heir to Hellenistic science since they had been part of the empire founded by Alexander the Great. Fortunately, they not only inherited this knowledge but they nurtured it while European culture stagnated through the Dark Ages.

The difficulty of depicting the advances experienced by the Arabs is attributable not only to the prejudices cited above, but also to the fact that most of the original material has been lost. Nevertheless, sufficient evidence is extant to ascertain considerable about the developments which took place.

Following Mohammed's ascendency, the initial scientific effort consisted primarily of translating the Greek and other sources into Arabic. However, numerous efforts in the advancement of science were being recorded. Probably the first of those who made substantial contributions was the philosopher Abu Ysuf Yaqub Ibn Is-haq (813-873), better known in the Western world by the Latinized name Al-Kindi. Born in Kufa, he later lived in Basra and Baghdad where his efforts ranged over broad aspects of physics.

Arabian influence on optics

Al-Kindi wrote *De Aspectibus* in which he dealt explicitly with optical problems. He asserted that vision had to take place by means of rays which promote a physical reaction upon the eye. This was in contradiction to the hypothesis advocated by Plato which considered that vision is accomplished by the expulsion of *ocular beams* from the eye. Al-Kindi attacked this concept, arguing that ocular beams are but mathematical abstractions which are incapable of acting physically or physiologically.

Al-Kindi extended the concept of a visual ray, noting that the formation of shadows suggests that light travels in straight lines. However, he was unable to explain how the rays could react within the eye to send necessary information to the soul permitting a visual reconstruction of the physical world. The concept of ocular beams thus remained widely accepted.

By the time of Al-Kindi's death, Arabian science was flourishing. Into this activity in A.H. 354 (A.D. 965) in Basra was born Abu Ali Mohammed Ibn al-Hasan Ibn al-Haitham, generally known in Arabian countries as Ibn al-Haitham and in Western countries by the Latinized Alhazen. I shall extend the courtesy here by referring to him as Ibn al-Haitham.

Although not much is known of his parents or of his early life, it can be inferred that he belonged to a middle-class family which was sufficiently well-to-do to provide him with an education, but not rich enough to provide him with the leisure to seek higher learning. He secured a position in a government office, suggesting that he must have done well in his studies and that his family exerted some local influence.

The government position provided al-Haitham with means for subsistence, but did not provide any intellectual stimulation. That was acquired during his spare hours by studying astronomy, mathematics, physics, and medicine. As his knowledge and self-confidence grew, he sought new horizons. He was naturally attracted to Egypt where the ruler, Fatimid Caliph Al-Hakim, was a patron of learning and had drawn several scholars to his court.

The Egyptian Court of Al-Hakim

Ibn al-Haitham reasoned that Al-Hakim would require some reason to invite him to Cairo. Thus, he made a thorough study of the Nile, which was and is the life blood of Egypt. The people of Egypt depend upon the water for irrigation, but dread the annual flood which causes considerable damage. Ibn al-Haitham concluded that a dam would entrap the water for the dry season and prevent the annual floods. He then prepared the outline of a plan to build such a dam at a site near Aswan and sent it to the Fatimid Caliph Al-Hakim. Naturally, Al-Hakim was impressed and invited Ibn al-Haitham to his court.

The date at which this occurred is uncertain even though, as we shall see, it is somewhat relevant. Al-Hakim was only eleven years old when he ascended the throne in 996. It seems unlikely that he would have achieved the maturity to invite al-Haitham to his court until perhaps a decade later. When he arrived in Cairo then, al-Haitham must have been at least in his early forties.

Proposed dam at Aswan

One of the first things that al-Haitham did upon his arrival was to visit Aswan. Prior to this he had not even visited Egypt, having made all of his plans for the dam from maps and studies available of the geography and geology of Egypt. With adequate funding al-Haitham traveled to Aswan, some 400 miles from Cairo. Here he made detailed surveys of the topography of the area, sampled the soil, noted the formation of rocks, and measured the width and discharge of the river. The river, incidentally, discharges about 17,000 cubic feet per second in the dry season and over 300,000 cubic feet per second after the rains.

Ibn al-Haitham soon realized that the task of constructing a dam was beyond the engineering capabilities of the time. This left two courses available to him. Either he could dilly-dally with the project, thus delaying the day of reckoning, or he could make a clean breast of the situation to the Caliph and hope for the best, knowing that Al-Hakim was short tempered and inclined to be cruel. Being a man of

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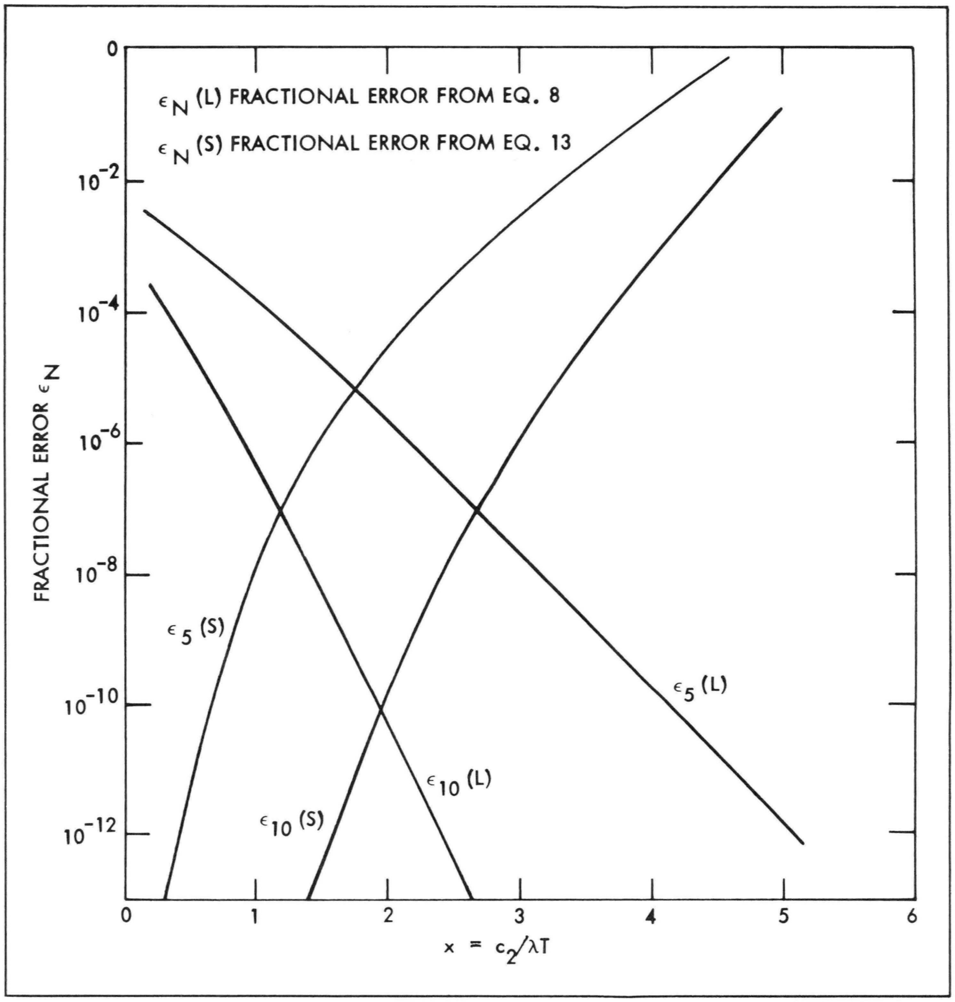

Figure 1. Fractional errors for two series approximations to the integral of the Planck function.

The use of the series of Eq. (4) to compute the integral over wavelength of the blackbody radiance is well known.[2,4] The use of the series of Eq. (11) to supplement these calculations when $x = c_2/\lambda T$ is small, is less well known. Its utility was, however, noted by Pivovonsky and Nagel.[5] More recently Ray Chandos used the approach described here to write hand-held calculator programs for integrals of Planck functions.[6] The two series combination very neatly overcomes the limitations of the single series, and is simple and efficient.

References

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4. *The Infrared Handbook*, W. L. Wolfe and G. J. Zissis, Eds., Office of Naval Research, Dept. of the Navy, Washington, D. C., 1978, Ch. 1.
5. M. Pivovonski and M. R. Nagel, *Tables of Blackbody Radiation Functions*, MacMillan, New York, 1961, p. xiv.
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As is clear from Bruce Emmons's discussion, he (as well as I and others in the past) associate the computational error related to the different methods of blackbody functions approximation to the bias of the estimate. The implicit assumption is that the values of the physical constants (c, h and k) are known exactly; however, an uncertainty exists for each of these constants. In a subsequent column, I will explore the implications of including the physical constants' uncertainties in several approximations to Planck's equation upon computation accuracy (i.e., bias and precision). I would also like to encourage you to express your thoughts on blackbody computations through this column.

R. Barry Johnson

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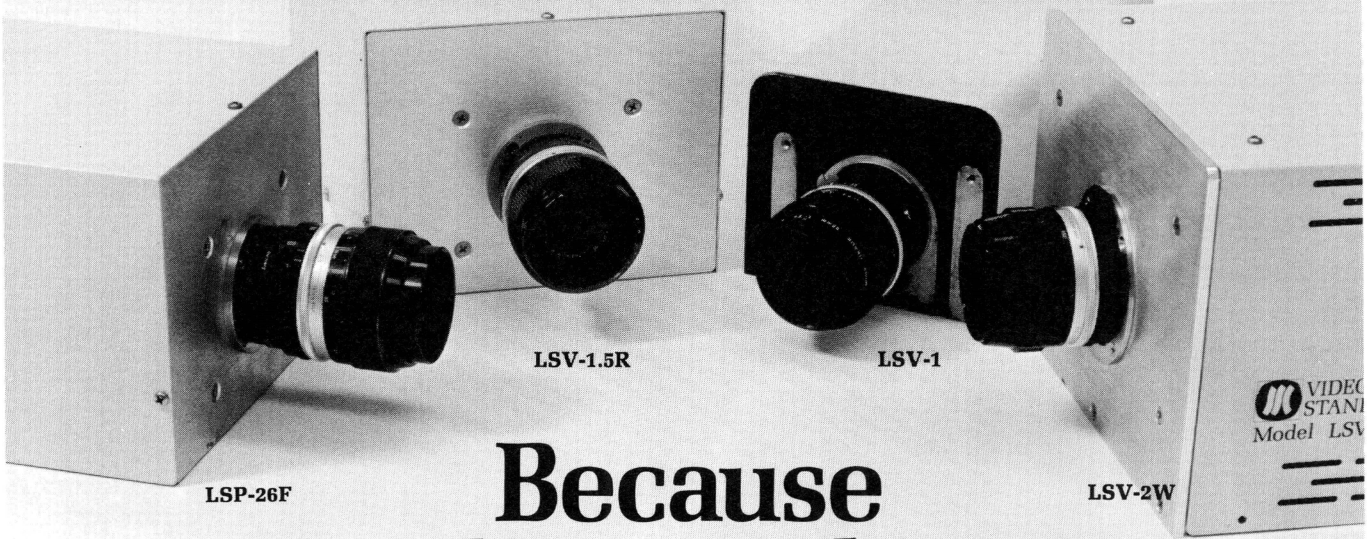

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high integrity, al-Haitham chose the latter route.

Fortunately, Al-Hakim suppressed his anger and passed no harsh order. But his subsequent behavior made it abundantly clear that Ibn al-Haitham had lost his favor and that the Caliph held a strong grudge against him. To avoid a harsher treatment, al-Haitham feigned madness. This was done so well that the Caliph was persuaded only to imprison al-Haitham and to confiscate all his books and scientific instruments.

Scientific pursuit

Again good fortune smiled upon optics, for Al-Hakim died in 1021, and al-Haitham was released. Since it is uncertain when he was imprisoned, we do not know how long he was deprived of his books and instruments, but certainly for an extended period al-Haitham had little to do but contemplate. At any rate, upon release he soon settled down to leading a scholar's life. Living near the University of Azhar, he earned sufficient money to meet his needs by copying such books as Euclid's *Elements of Geometry* and Ptolemy's *Almagest*.

Ibn al-Haitham thus plunged into the passion of his life—scientific pursuit. At an age in his mid-fifties he began valuable contributions to geometrical and physiological optics which continued for over two decades. The results of his optical researches were recorded in his *Kitab-ul-Manazir* (Treatise on Optics).

This was done during a period when there was still a strong submission to the writings of the ancient scholars. Aristotle was referred to, even by al-Haitham, as *The Master* and his work was considered inviolable. However, al-Haitham considered that experimentation was necessary to the understanding of a phenomenon.

One of al-Haitham's first contributions consisted of rejecting Plato's ocular beams. He argued that the apparent sizes of objects viewed at a distance are easier explained by rays of light emitted from the object than upon ocular beams.

Through experimentation, al-Haitham divided transparent bodies into two classes: "celestial" and "sub-celestial." The former is absolutely transparent and became the forerunner of the concept of the ether. The sub-celestial bodies were divided into three sub-categories, consisting of gases, fluids, and solids. He maintained that the propagation of light through a transparent body is a physical characteristic of all kinds of light rather than a characteristic of the body.

Ibn al-Haitham studied the phenomena of reflection and refraction, showing experimentally that the incident ray, the reflected (or refracted) ray, and the surface normal all lie in a plane. His method is often used today to illustrate this. He attempted to quantify the law of refraction, but could only show that the ratio of the angle the incident ray makes with the surface normal to that made by the refracted ray is about 1.3 for angles less than 20 degrees. Although Muslim mathematicians had worked out the concept of the sines of angles, it would be about 600 years before the correct law of refraction was stated.

Ibn al-Haitham's studies included the formation of images by spherical and parabolic mirrors. His pioneering efforts in this field were accepted as the standard work for several centuries. He also made a pinhole camera, and described how an inverted image of a candle is formed thereby.

Physiological optics

Perhaps his greatest contributions to optics lie in his efforts in physiological optics. He described the various parts of the eye and their function. The opaque coating of the eye outside the iris which forms the white of the eye and is known as the sclerotic was discussed. The function of the horny transparent structure in front of the eyeball, known as the cornea, was studied and reported. He also investigated the characteristics of the membrane

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behind the cornea (the choroid), the iris, the aqueous humor, and the retina. His description of the eye has been termed masterly and remains as the basis for description today. Several of the terms he used have been literally translated into Latin and are used today. For instance, the lens of the eye suggested to al-Haitham a certain grain, so he used the phrase *adasa* which is the Arabic word for this grain. This particular grain is known as *lentil* in Latin, and thus *adasa* was translated to lens. It is interesting to note in passing that literal translations sometimes result in amusing results. For instance, the English word *airline* translates into an Arabic word suggesting geometrical lines in the sky!

Ibn al-Haitham also described the formation of a halo, the scattering of light by dust particles in a darkened room, the duration of twilight, and similar phenomena.

At the age of 74, al-Haitham died. Although Westerners may be unaware of his life, he has been called *the greatest authority on optics in the Middle Ages*. His work is said to have had a profound influence on Roger Bacon, Wittelo, Leonardo da Vinci, Johann Kepler, and Sir Isaac Newton. And this was accomplished by an Arabian scholar after he was released from prison at an age greater than 50.

MEDICINE VIII
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Book Reviews

CUTTING AND POLISHING OF ELECTRO-OPTIC MATERIALS, G. W. Fynn and W. J. A. Powell. Illustrations, line drawings, extensive tables, index. ISBN 0-470-26607-4. Halsted Press, John Wiley and Sons, New York (1979) $67.50.

Reviewed by W. P. Barnes, Jr., Optical Systems Division, Itek Corporation, 10 Maguire Road, Lexington, MA 02173.

Fynn and Powell and the Halsted Press have produced a valuable addition to the practical literature on optical component fabrication. This conclusion is reached after fairly extensive sampling, with a growing conviction that a real appreciation and useful exploitation of this work will best follow from a complete reading, then restudy as one encounters opportunities to use the operations described.

Serial reading of at least whole chapters is recommended—the level of detail required, and given, is such that a sentence skipped may invite failure. Fortunately, the text is eminently readable and one can give it close attention if it is not taken in too large single doses. The text is also well supported by extensive use of photographs, diagrams, and tabular summaries. All in all, a fine example of well-executed technical exposition.

The authors have limited themselves to items within their own experience. Interpolation of their specific descriptions to other equipment and the differences between UK and USA (or other) suppliers should pose no real difficulties. They also apologize somewhat for semi-scientific solutions rather than rigorous investigation and explanation. Rigor is commendable, but often impractical, while for much of our optical industry even semi-science is a great leap forward.

Their final prefatory caution, "the expositions may be simple but their executions seldom are," recognizes that craftsmanship remains essential. Their text is however, an ample illustration that the processes for cutting and polishing electro-optic materials are *not* inenarrable.

Two areas of interest and recent rapid development lie beyond the boundaries Fynn and Powell imposed on themselves. These are quantitative evaluation of surface microfinish, by scattered light and precision profilometry, and the growing use of lasers for cutting, scribing, and marking operations.

This book should be of value to all manufacturing engineers, supervisors, opticians and technicians concerned with optical materials other than glass. I believe this book will be found useful, and recommend that it be found in the shop, not the library.