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## **Generic algorithm of phase reconstruction in phase-shifting interferometry**

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**Abstract.** A generic reconstruction algorithm is proposed, which is obtained by using a phase-shifting technique with an arbitrary number of phase shifts between intensity measurements. The generic algorithm entirely describes the structure of known phase-shifting algorithms and permits the construction of new ones with an arbitrary number of phase shifts. © 2013 Society of Photo-Optical Instrumentation Engineers (SPIE) [DOI: 10.1117/1.OE.52.3.030501]

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## 1 Introduction

Phase-shifting interferometry methods have been widely used in optical interference systems in recent years. Widespread use of these methods has resulted from the simplicity of specifying the values of phase shifts, the low complexity of the algorithms, and the high precision that they can achieve. At the same time, the layouts of interferometers can be easily modified.

A large number of expressions for phase reconstruction with an arbitrary number of phase shifts are known. The earliest algorithms used decoding equations with three or four shifts. With an increased amount of available computational power, it became possible to use algorithms with a larger number of shifts. Thus, in Ref. 1, an algorithm that uses 15 phase shifts was introduced, and an algorithm using 101 phase shifts was described in Ref. 2.

The equations used for the reconstruction are derived by solving trigonometric equations. In Ref. 3, an analysis of formulas for phase reconstruction with an arbitrary number of the phase shifts is given, but the number of shifts should be constant. In this work, we propose a generic algorithm that allows the structure of known algorithms to be determined and the construction of new algorithms with an unlimited number of arbitrary, and not always constant, phase shifts.

## 2 Synthesis of the Algorithm

Phase-shifting methods are based on the capture of several interferograms while the phase of the reference wave continues to follow the specified values. At different phase shifts, the intensity of an interferogram with phase shift  $\delta_i$  can be represented as

$$I_i(x, y) = A(x, y) + B(x, y) \cos(\phi(x, y) + \delta_i), \quad (1)$$

where  $A(x, y)$  and  $B(x, y)$  are the average brightness of the interferogram and the amplitude of the interference pattern in the point  $(x, y)$ , respectively,  $i = 0, 1, 2, \dots, m - 1$ ,  $m$  is the number of phase shifts, and  $\delta_0 = 0$ .

The algorithms obtained at  $m$  different values are called  $m$ -point algorithms. There are many algorithms that work for different numbers of phase shifts. Many of the possible implementations arose from an interest in determining a generic scheme for the algorithms. A generic scheme allows evaluation of the pros and cons of a specific variant and determining the methodology of the algorithm construction.

Equation (1) can be represented in vector form as

$$\vec{I} = \vec{A} + (B \cos \phi) \vec{C} - (B \sin \phi) \vec{S}, \quad (2)$$

where  $\vec{I}$  is the set of intensities for different phase shifts  $\delta_i$  at each point of the interferogram  $I_i(x, y)$ ,  $\vec{A} = A \cdot (1, \dots, 1)^T$  is an  $m$ -dimensional vector,  $\vec{C} = (\cos \delta_0, \dots, \cos \delta_{m-1})^T$ ,  $\vec{S} = (\sin \delta_0, \dots, \sin \delta_{m-1})^T$ , and the size of the vectors is determined by the number of phase shifts.

Let us rewrite Eq. (2) as follows:

$$\vec{I} \cdot \vec{C}^\perp = \vec{A} \cdot \vec{C}^\perp + (B \cdot \cos \phi) \vec{C} \cdot \vec{C}^\perp - (B \cdot \sin \phi) \vec{S} \cdot \vec{C}^\perp \quad (3a)$$

and

$$\vec{I} \cdot \vec{S}^\perp = \vec{A} \cdot \vec{S}^\perp + (B \cdot \cos \phi) \vec{C} \cdot \vec{S}^\perp - (B \cdot \sin \phi) \vec{S} \cdot \vec{S}^\perp. \quad (3b)$$

To extract the quadrature components  $\sin \phi$  and  $\cos \phi$ , we can use a property of the dot product for orthogonal vectors ( $a \cdot a^\perp = 0$ ). Let  $\vec{C}^\perp$  be the result of the cross product  $\vec{A}$  and  $\vec{C}$ ,  $\vec{C}^\perp = \vec{C} \times \vec{A}$ , and  $\vec{S}^\perp$  be the result of the cross product  $\vec{A}$  and  $\vec{S}$ ,  $\vec{S}^\perp = \vec{S} \times \vec{A}$ . Taking into account that  $\vec{S}^\perp$  is orthogonal to  $\vec{A}$  and  $\vec{S}$ , and the mentioned property of the dot product, we obtain

$$\vec{I} \cdot \vec{C}^\perp = -(B \cdot \sin \phi) \vec{S} \cdot \vec{C}^\perp \quad (4a)$$

and

$$\vec{I} \cdot \vec{S}^\perp = (B \cdot \cos \phi) \vec{C} \cdot \vec{S}^\perp; \quad (4b)$$

then

$$B \sin \phi = -\frac{\vec{I} \cdot \vec{C}^\perp}{\vec{S} \cdot \vec{C}^\perp} \quad (5)$$

and

$$B \cos \phi = \frac{\vec{I} \cdot \vec{S}^\perp}{\vec{C} \cdot \vec{S}^\perp}. \quad (6)$$

Considering the properties of the dot product and the cross-product  $c(b \times a) = -b(c \times a)$  in the case of the non-cyclic permutation of the vectors, we obtain

$$(\vec{S} \cdot \vec{C}^\perp) = \vec{S}(\vec{C} \times \vec{A}) = -\vec{C}(\vec{S} \times \vec{A}) = -(\vec{C} \cdot \vec{S}^\perp).$$

Then, a reconstruction formula can be represented in the vector form as

$$\frac{\sin \phi}{\cos \phi} = \frac{\vec{I}^\perp \cdot \vec{C}}{\vec{I}^\perp \cdot \vec{S}}, \quad \phi = \arctan\left(\frac{\vec{I}^\perp \cdot \vec{C}}{\vec{I}^\perp \cdot \vec{S}}\right), \quad (7)$$

because in this case only the vector  $\vec{I}^\perp$  is calculated. For the case of three shifts,

$$\begin{aligned} \vec{I}^\perp &= \vec{I} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = M \cdot \vec{I} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \cdot \vec{I} \\ &= \begin{bmatrix} I_2 - I_3 \\ I_3 - I_1 \\ I_1 - I_2 \end{bmatrix}, \end{aligned} \quad (8)$$

where  $M$  is a matrix, which calculates the cross-product of the vectors.<sup>4</sup> Then, we obtain the equation described in Ref. 5:

$$\phi = \arctan \frac{(I_2 - I_3) \cos \delta_1 + (I_3 - I_1) \cos \delta_2 + (I_1 - I_2) \cos \delta_3}{(I_3 - I_2) \sin \delta_1 + (I_1 - I_3) \sin \delta_2 + (I_2 - I_1) \sin \delta_3}. \quad (9)$$

If  $\delta_1 = \pi/4$ ,  $\delta_2 = 3\pi/4$ , and  $\delta_3 = 5\pi/4$ , we obtain the expression described in Ref. 6,

$$\phi = \arctan \frac{I_3 - I_2}{I_1 - I_2}. \quad (10)$$

The  $m$ -dimensional matrix  $M$  ( $m \geq 3$ ) can be presented as

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}_{m \times m}. \quad (11)$$

and for the case of four shifts,

$$\vec{I}^\perp = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} I_2 - I_4 \\ -I_1 + I_3 \\ -I_2 + I_4 \\ I_1 - I_3 \end{bmatrix}. \quad (12)$$

If  $\delta_1 = 0$ ,  $\delta_2 = \pi/2$ ,  $\delta_3 = \pi$ , and  $\delta_4 = 3\pi/2$ , we obtain the expression found in Ref. 7:

$$\phi = \arctan \frac{I_4 - I_2}{I_1 - I_3}. \quad (13)$$

Obtaining Eqs. (9), (10), and (13) is not necessary if a generic algorithmic procedure that implements Eq. (7) is used:

$$\phi = \arctan \frac{\sum_{i=0}^{m-1} [(I_{\text{mod}(i+1,m)} - I_{\text{mod}(m+i-1,m)}) \cdot (\delta_i)]}{\sum_{i=0}^{m-1} [(I_{\text{mod}(i+1,m)} - I_{\text{mod}(m+i-1,m)}) \cdot \sin(\delta_i)]}, \quad (14)$$

where  $\text{mod}(i, m)$ , the remainder when  $i$  is divided by  $m$ .

### 3 Conclusions

We have proposed an algorithm that can be used to obtain reconstruction formulas without solving systems of trigonometric equations. Our procedure can be used for any set of arbitrary phase shifts. Note that the proposed method allows corresponding real, but not assumed, values of phase shifts to be used in the reconstruction formulas. This property can undoubtedly help to increase the precision of phase calculation in phase-shifting interferometry.

### References

1. P. de Groot, "Phase-shift calibration errors in interferometers with spherical Fizeau cavities," *Appl. Opt.* **34**(16), 2856–2863 (1995).
2. P. de Groot, "101-frame algorithm for phase shifting interferometry," *Proc. SPIE* **3098**, 283–292 (1997).
3. P. A. A. Magalhães et al., "New equations for phase evaluation in measurements with an arbitrary but constant phase shift between captured intensity signs," *Opt. Eng.* **48**(11), 113602 (2009).
4. S. Liu and G. Trenkler, "Hadamard, Khatri-Rao, Kronecker and other matrix products," *Int. J. Inform. Syst. Sci.* **4**(1), 160–177 (2008).
5. P. Hariharan, B. F. Oreb, and N. Brown, "Digital phase-measurement system for real-time holographic interferometry," *Opt. Commun.* **41**(6), 393–398 (1982).
6. J. C. Wyant and K. Creath, "Recent advances in interferometric optical testing," *Laser Focus* **21**(11), 118–132 (1985).
7. J. C. Wyant, "Interferometric optical metrology: basic system and principles," *Laser Focus* **18**(5), 65–67 (1982).