ICSO 2016

International Conference on Space Optics

Biarritz, France

18–21 October 2016

Edited by Bruno Cugny, Nikos Karafolas and Zoran Sodnik

Toward optimum efficiency in a quantum receiver for coded ppm

D. M. Boroson

esa icso proceedings cnes

International Conference on Space Optics — ICSO 2016, edited by Bruno Cugny, Nikos Karafolas,
Zoran Sodnik, Proc. of SPIE Vol. 10562, 105623M · © 2016 ESA and CNES
CCC code: 0277-786X/17/$18 · doi: 10.1117/12.2296040

TOWARD OPTIMUM EFFICIENCY IN A QUANTUM RECEIVER FOR CODED PPM

D. M. Boroson
USA

I. INTRODUCTION

Communications systems builders continue to search for signal formats and receiver architectures that can provide the most efficient utilization of their subsystems, which include power amplifiers as well as transmit and receive apertures. Receivers requiring very small amounts of received power are of particular interest in communications links where transmission distances are very long and losses are large, such as from Deep Space.

Helstrom and others ([1],[2],[3]) initiated the study of optimum signal reception using quantum mechanical signal models. They derived the mathematical description and predicted performance of receivers that optimize certain criteria, such as Minimum Probability of Error (MPE). Unfortunately, practical implementation of their proposed receivers has still not been achieved.

In parallel, technology has advanced to where noiseless photon counters can be used to achieve quite good performance ([4]). We show here that, when an end-to-end error correction code is added, in fact such a system can out-perform the "optimum" MPE system at low signal powers.

In this report, we derive the formulation of a quantum receiver that is shown to be uniformly better than either the MPE or photon-counting receiver**.**

II. PPM SIGNAL DESCRIPTION

We will examine the set of signals known as Pulse Position Modulation (PPM), although the results we will show are valid for any set of signals that are classically orthogonal. Source bits are grouped into K-bit symbols. (An example is shown in Figure 1.) The time duration of the K bits is then divided into $M=2^K$ shorter time slots, and the laser is turned on only in that slot corresponding to the particular K bits. When a transmitter has the property that it is average power limited (such as in doped-fiber optical amplifiers) then the power in the single "on" slot is M times higher than the average.

This signal is transmitted and received, after much loss, at the distant receiver. Thus, the PPM signal's quantum description is a set of pure states,

$$\left\{\left|\psi_0\right\rangle,\left|\psi_1\right\rangle,\left|\psi_2\right\rangle\ldots\ldots\left|\psi_{M-1}\right\rangle\right\} \tag{1.}$$

each the tensor product of M pure, coherent states ([5]). Some examples from an M=8 signal set are shown here:

$$\left|\psi_0\right\rangle=|\alpha\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle \quad , \left|\psi_4\right\rangle=|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|\alpha\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle \tag{2.}$$

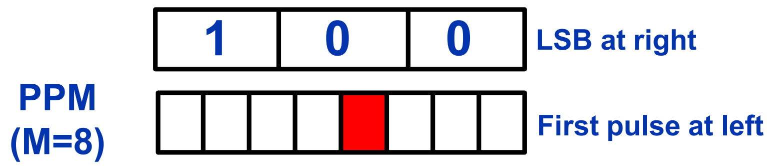


Figure 1. 8-ary PPM example; data =100 = 4

where each coherent state is infinite-dimensional,

$$|\alpha\rangle = \exp\left(\frac{-|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \quad , |\alpha|^2 = N_S \text{ is the average number of photons per symbol.} \tag{3.}$$

We assume the highly simplified model with no extraneous noises. We can note that these M states are linearly independent and span only an M-dimensional subspace of the much larger Hilbert space. It can be shown that this signal set has a high degree of symmetry, known as "geometrically uniform" symmetry, ([6], [7]) and we assume equal prior probabilities.

II. KNOWN RECEIVERS

A. Minimum Probability of Error Receiver

The basics of quantum-optimum receiver structures were pioneered by Helstrom, Yuen, Holevo, and others ([1],[2],[3]). The first metric they used for optimization was Minimum Probability of Error (MPE), when receiving a symbol. It was shown that the receiver was a Positive Operator-Valued Measure (POVM) and that, when the M signals were pure, it had the particularly simple form of a set of M orthonormal projectors. We denote these projectors

$$\{|\eta_0\rangle, |\eta_1\rangle, |\eta_2\rangle \ldots\ldots |\eta_{M-1}\rangle\} \tag{4.}$$

and note that the outputs have probabilities according to

$$P[k \mid m] = |\langle \eta_k | \psi_m \rangle|^2 \tag{5.}$$

Optimization can thus be performed by finding the set of inner products, $\{\langle \eta_k | \psi_m \rangle\}$ between the state vectors and the projectors, making sure to preserve both the known inner products of the various signal states and the orthonormality of the projectors.

A geometric view of signals and projectors is shown in Figure 2 for M=3.

It has also been shown ([6], [7]) that the MPE optimization is achieved in geometrically uniform systems with the so-called Square Root Measurement, which has a closed form for the projectors and the performance. (We will not reproduce that material here.)

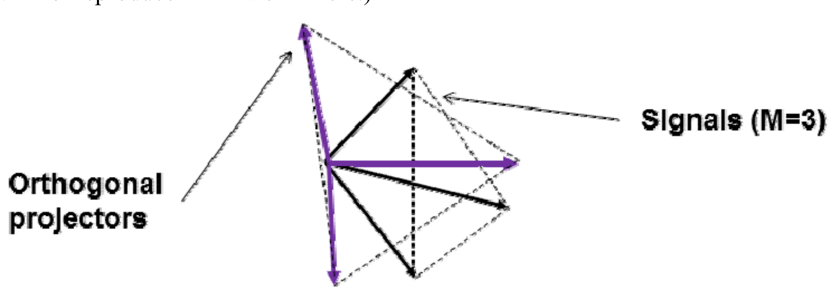


Figure 2. Geometric view of 3-ary PPM signals with the POVM.

For our (classically) orthogonal and geometrically uniform signal set, Helstrom ([8]) showed that one can also find the MPE solution by parameterizing the solution with an MxM matrix X where

$$X_{km} = \langle \eta_k | \psi_m \rangle . \tag{6.}$$

From the known symmetry, we can let (following Helstrom)

$$a = X_{kk} \text{ and } b = X_{km}, \text{ k} \neq \text{m} \tag{7.}$$

To preserve orthonormality, we find

$$a^2 + (M-1)b^2 = 1 . \tag{8.}$$

and from the known correlations of the coherent signal states,

$$\langle \psi_k | \psi_m \rangle = \begin{cases} 1, k = m \\ \xi \equiv \exp(-N_S), k \neq m \end{cases} \tag{9.}$$

we find

$$2ab + (M-2)b^2 = \xi . \tag{10.}$$

These can be solved for a and b, (still following Helstrom)

$$a = \frac{1}{M}\left\{[1+(M-1)\xi]^{1/2} + (M-1)[1-\xi]^{1/2}\right\},\ b = \frac{1}{M}\left\{[1+(M-1)\xi]^{1/2} - [1-\xi]^{1/2}\right\} \tag{11.}$$

where the symbol error probability can be shown to be $1 - a^2 = (M-1)b^2$.

B. Unambiguous Receiver

Some authors have investigated, instead, receivers where at least one of the outputs is promised to be perfectly correct ([9], [10], [11], [12]). One can achieve this (in our noiseless system) by either selecting one or more of the signals to favor, or, in our preferred symmetric treatment, by treating the states symmetrically similar but allowing an extra output corresponding to an ambiguous measurement. Such a system has M+1 outputs and has been described in the general case ([13]). Its POVM operates in an M+1-dimensional Hilbert subspace.

We note that this quantum optimum (for this new metric) receiver has performance equivalent to the well-known noiseless photon-counting receiver ([14]) where one or more photon counts correctly tells us which PPM pulse was sent while the lack of any photons corresponds to an ambiguous measurement. Classically, we call such an ambiguous measurement an erasure. In a photon-counted (as well as unambiguous quantum) PPM system, an erasure occurs with probability

$$P_{erasure} = 1 - \exp(-N_S) . \tag{12.}$$

II. CODED COMMUNICATIONS

One needs to decide what to do with the information that an erasure has occurred. When sending data from one party to another, knowledge that some data has been erased has at least two uses:

1) The receiver can request a retransmission

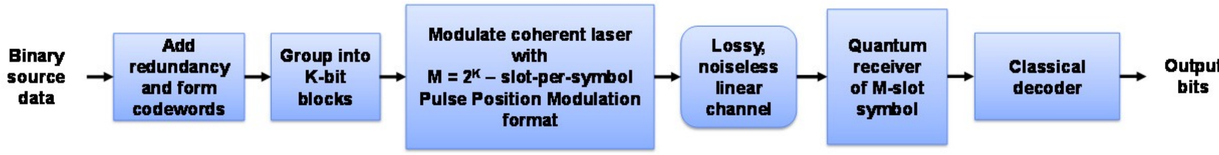


Figure 3. Coded communications system with semi-classical receiver architecture.

2) The system can employ an end-to-end code, which includes redundant bits, whose decoder can use the knowledge of the erasure to help recreate the original bits efficiently. (For instance, Reed-Solomon codes have a well-known decoding algorithm for Errors and Erasures.)

Such a system can be described as in the block diagram in Figure 3. A receiver that uses a quantum system to make symbol decisions, but then uses the outputs in a classical decoder is sometimes called "semi-classical." Its performance can be assessed by calculating the "accessible information" in the symbol receiver outputs, and then applying a near-(Shannon)-capacity-achieving code, many of which are now known. This information is calculated using the well-known formula (for our equal-priors system)

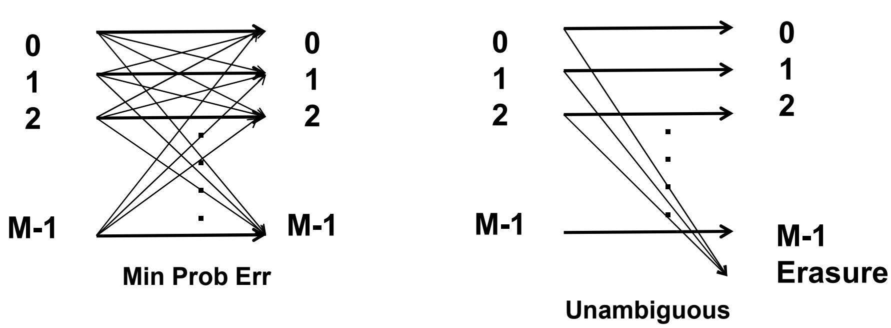
$$I \equiv \frac{1}{M}\sum_{i,j} P[j\,|\,i]\log\frac{P[j\,|\,i]}{P_y[j]}, \text{ where } \quad P_y[k] = \frac{1}{M}\sum_m P[k\,|\,m] \tag{13.}$$

The two channels that we have investigated can be described with the graphs shown in Figure 4. Using the transition probabilities we have investigated, we can calculate and plot the accessible information. An example of M=16 is shown in Figure 5a. It is clear that the unambiguous receiver actually performs better than the "optimum" MPE at low signal levels. Thus, the benefit of adding an extra output state becomes clear. We can show the same results perhaps more clearly by showing the photon efficiency of our receiver assuming a capacity-achieving code, and plotting it versus the extra bandwidth required by the PPM signaling plus the capacity-achieving code. The M=16 example is shown in Figure 5b. Here we can see that the unambiguous system performs more than a dB better at very low code rates, although the MPE system is slightly better at high code rates.

One can ask, what is the optimum symbol receiver for a semi-classical system? Holevo ([15]) investigated this problem, and derived a necessary condition for a receiver to be information-optimum. We can write his condition, using our X-matrix notation, as

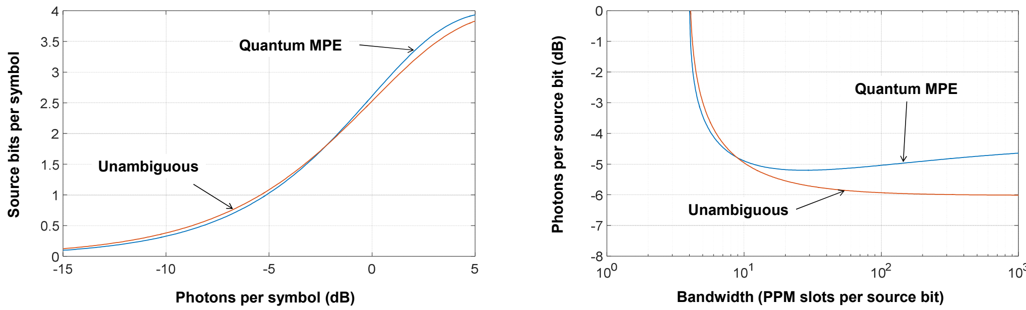
$$\sum_i X_{ji}\overline{X}_{ki}\log\left[\frac{|X_{ji}|^2\sum_r |X_{kr}|^2}{|X_{ki}|^2\sum_s |X_{js}|^2}\right] = 0 \quad \text{for all j, k} \tag{14.}$$

This formula does not seem to allow closed-form solution, nor even suggest the required dimensionality of the outputs. We can use it, though, to check whether a proposed system at least satisfies the necessary condition. (As a note, the MPE satisfies this equation.)



(a) (b)

Figure 4. (a) – MPE transitions; (b) – Unambiguous receiver transitions



(a) (b)

Figure 5. (a) Accessible information for MPE and Unambiguous receivers for 16-PPM. (b) Same information plotted as photon efficiency vs bandwidth expansion when using a capacity-achieving code.

Davies ([16]) also addressed this problem. Although he did not find the optimum receiver (nor its performance), he bounded the number of measurements, k, that would be required to achieve the maximum information. He showed that, when operating in a Hilbert space of finite dimension d, that

$$d \le k \le d^2 \qquad (15.)$$

Sasaki etal ([17]) refined this result by noting that if the Hilbert space is real (as in our PPM system, although our coherent states have infinite dimension) that the upper bound can be reduced to be

$$d \le k \le d(d+1)/2 \qquad (16.)$$

In classical receivers, a measurement with more bits describing it than just the symbol hard decision is called a "soft decision." Many modern decoders can make use of such information.

It is of interest that Levitin ([18]) proved that, for M=2, two measurements suffice and that the MPE is the optimum form for optimizing information with pure states.

III. A NEW PPM RECEIVER

A. An Errors and Erasures Receiver

We are motivated to find a symbol-wise receiver which has, by symmetry, either rM or rM+1 outputs for some integer r, where the second option would include an erasure output. We will examine r=1. The transition graph for this general case of M+1 outputs is depicted in Figure 7. We see that we are now allowing both errors and erasures.

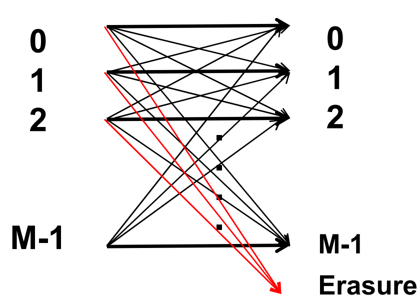


Figure 7. A quantum Errors and Erasures receiver with M+1 outputs for the M-ary PPM system.

Observing the symmetry, we can create the matrix of state/projector inner products

$$X = \begin{bmatrix} e & e & e & e & \dots \\ a & b & b & b & \dots \\ b & a & b & b & \dots \\ b & b & a & b & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{17.}$$

where $P_{erasure} = e^2$. The two equations to be solved are thus

$$a^2 + (M-1)b^2 + e^2 = 1, \quad 2ab + (M-2)b^2 + e^2 = \xi \tag{18.}$$

These can be solved, keeping e as a free parameter, as

$$a = \frac{1}{M}\left\{[1+(M-1)\xi - Me^2]^{1/2} + (M-1)[1-\xi]^{1/2}\right\}, b = \frac{1}{M}\left\{[1+(M-1)\xi - Me^2]^{1/2} - [1-\xi]^{1/2}\right\} \tag{19.}$$

where we see that we must have $e^2 \le \frac{1+(M-1)\xi}{M}$

We have substituted these into (13), the equation defining accessible information, and numerically optimized by selecting e. In Figure 8, we show both e^2, the optimized erasure probability, and $(M-1)b^2$, the error probability for the M=16 case. We note that, as expected, at low flux, the receiver looks more like the photon-counting, unambiguous receiver. We also see that there is a threshold flux above which the MPE is the best such receiver, and erasures are not needed.

The accessible information using this receiver has been plotted in Figure 9a, along with that of the MPE and the unambiguous receiver. We see that this Errors and Erasures receiver is better than either of the previous receivers, although the improvement is small, about 0.3 dB at rate ½. The improvement is more apparent when plotted in the photon efficiency curve of Figure 9b, where we can see that the addition of errors improves the unambiguous receiver at low code rates.

B. The Quantum Optimum

For completeness, we need to note that the truly optimum receiver for our coded system would find a quantum optimum receiver for the full codeword, most likely by implementing the decoder in a quantum computer. Unfortunately, no one has yet invented a structured format for such a receiver. We do know the performance it could achieve, however ([19]) which can be shown to be

$$I = \log(M) - \frac{1}{M}\left\{[1+(M-1)\xi]\log[1+(M-1)\xi] + (M-1)[1-\xi]\log[1-\xi]\right\}. \tag{20.}$$

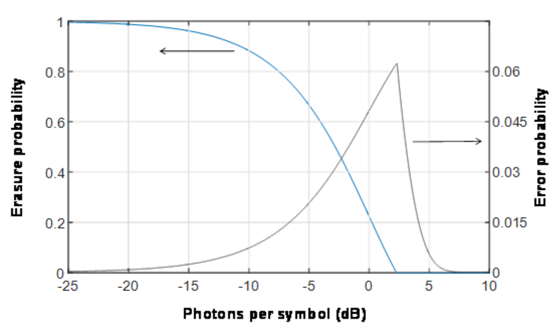
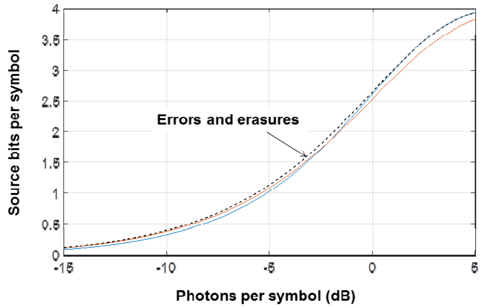
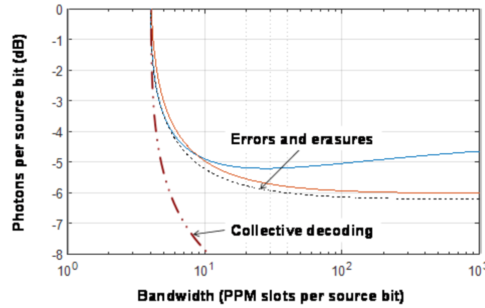


Figure 8. Numerically optimized error and erasure probabilities for 16-ary quantum Errors and Erasures receiver.



(a)



(b)

Figure 9. (a) Accessible information for Errors and Erasures receiver, along with MPE and unambiguous receivers. (b) Same information plotted as photon efficiency vs bandwidth expansion when using a capacity-achieving code. Also shown is the quantum optimum, achieved with collective quantum decoding of the full codeword.

This has also been plotted in Figure 9b. (See [20].) We can see that such a fully optimum quantum receiver would be quite a bit better than even our improved semi-classical receiver.

IV. IMPLEMENTATION POSSIBILITIES

Of these several types of receivers, it is only the unambiguous one that has a straightforward implementation. That is, high-quality photon-counting devices are an excellent means of noiselessly detecting photons (or the lack of photons.) (See [21] for an example of such a system.)

Both the MPE and Errors and Erasures receivers, however, require superposition states, and are thus much more difficult to implement. A means of mapping coherent signals into a quantum computer was presented in [22], and such an approach could be used to achieve either of these POVM systems. For the new receiver, it might be possible instead to use a Quantum Non-Demolition measurement as a first stage ([23]) to deduce whether there were any photons at all (ie, that there was not an erasure) followed by an MPE-like POVM, although the QND receiver is known to randomly change the phase of the remaining signal, which may be found to erase any potential gains.

SUMMARY

For communications systems allowing quantum-optimum symbol receivers plus classical decoders, which are sometimes called semi-classical coded systems, we have shown how an extension of the standard MPE receiver to include errors as well as ambiguous (ie, erasure) outputs can increase the accessible information available over either MPE receivers or photon-counting receivers. Future efforts may well find more general extensions to improve performance further.

REFERENCES

[1] C. W. Helstrom, "Detection theory and quantum mechanics," *Inf. Control*. **10**, 254–291 (1967)
[2] A. S. Holevo, "Statistical decision theory for quantum systems," *J. Multivariate Anal*. **3**, 337–394 (1973)
[3] H. P. Yuen, R. S. Kennedy, and M. Lax, "Optimum testing of multiple hypotheses in quantum detection theory," *IEEE Trans. Inf. Theory* **IT-21**, 125–134 (1975)
[4] M. E. Grein, A. J. Kerman, E. A. Dauler, M. M. Willis, B. Romkey, R. J. Molnar, B. S. Robinson, D.V. Murphy, and D. M. Boroson, "An Optical Receiver for the Lunar Laser Communication Demonstration Based on Photon-Counting Superconducting Nanowires," *Proc. SPIE 9492*, Advanced Photon Counting Techniques IX, 949208 (May 28, 2015)
[5] G. Cariolaro, G. Pierobon, "Theory of Quantum Pulse Position Modulation and Related Numerical Problems," *IEEE Trans Comm*, VOL. 58, NO. 4, April 2010, pp 1213-1222.
[6] P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, and W. K. Wootters. "Classical Information Capacity of a Quantum Channel." *Physical Review A* 54, no. 3 (1996): 1869.
[7] Y.C. Eldar, and G. D. Forney Jr. "On Quantum Detection and the Square-Root Measurement." *Information Theory, IEEE Transactions on*, 47, no. 3 (2001): 858–872.

[8] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, 1976)
[9] I. D. Ivanovic, "How to differentiate between non-orthogonal states," *Phys. Lett. A* **123**, 257–259 (1987)
[10] D. Dieks, "Overlap and distinguishability of quantum states," *Phys. Lett. A* **126**, 303–306 (1988)
[11] A. Peres, "How to differentiate between non-orthogonal states," *Phys. Lett. A* **128**, 19–19 (1988)
[12] G. Jaeger and A. Shimony, "Optimal distinction between two non-orthogonal quantum states," *Phys. Lett. A* **197**, 83–87 (1995)
[13] A. Chefles, "Unambiguous discrimination between linearly independent quantum states," *Phys. Lett. A* **239**, 339–347 (1998)
[14] J. Pierce, E. Posner, and E. Rodemich. "The Capacity of the Photon Counting Channel." *IEEE Transactions on Information Theory* 27, no. 1 (1981): 61–77.
[15] A. S. Holevo, "Statistical decision theory for quantum systems," *J. Multivariate Anal*. 3, 337–394 (1973
[16] E. B. Davies, "Information and quantum measurement," *IEEE Trans. Inf. Theory* IT-24, 596–599 (1978)
[17] M. Sasaki, S. M. Barnett, R. Jozsa, M. Osaki, and O. Hirota, "Accessible information and optimal strategies for real symmetrical quantum sources," *Phys. Rev. A* **59**, 3325–3335 (1999).
[18] L. B. Levitin, "Optimal quantum measurements for two pure and mixed states," in *Quantum Communications and Measurement*, V. P. Belavkin, O. Hirota, and R. L. Hudson, eds. (Plenum, 1995), pp. 439–448.
[19] A. S. Holevo. "Quantum Coding Theorems." *Russian Mathematical Surveys* 53, no. 6 (1998): 1295–1331.
[20] D. M. Boroson, "Optical Communications: A Compendium of Signal Formats, Receiver Architectures, Analysis Mathematics, and Performance Comparisons," Report 60-1054, M.I.T. Lincoln Lab., Lexington, Mass., 9 Sept. 2005, http://oai.dtic.mil.
[21] D. M. Boroson, B. S. Robinson, "The Lunar Laser Communication Demonstration: NASA's First Step Toward Very High Data Rate Support of Science and Exploration Missions," *Space Science Reviews* 185(1-4), December 2014.
[22] M. P. da Silva, S. Guha, and Z. Dutton. "Achieving Minimum-Error Discrimination of an Arbitrary Set of Laser-Light Pulses." *Physical Review A* 87, no. 5 (May 23, 2013).
[23] Imoto, N., H. A. Haus, and Y. Yamamoto. "Quantum Nondemolition Measurement of the Photon Number via the Optical Kerr Effect." *Physical Review A* 32, no. 4 (1985): 2287.