Thresholds and Noise

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ABSTRACT

Random processes acting through dynamical systems with thresholds lie at the heart of many natural and man-made phenomena. The thresholds here considered are general including not only sharp or "hard" boundaries but also a class of dynamical, nonlinear system functions some of which are themselves mediated by the noise. Processes include noise-induced transitions, postponed and advanced bifurcations, noise enhanced propagation of coherent structures, and stochastic resonance and synchronization. Examples of these processes are found in a wide range of disciplines from physics and chemistry to neuroscience and even human and animal behavior and perception. I will discuss some of these examples connecting them with their fundamental dynamical origins

Keywords: Threshold, noise, bifurcation, postponement, noise-induced transition, stochastic resonance, synchronization, coherence

1. INTRODUCTION

Noise has become the common word used to describe a variety of random processes usually operative within or on a dynamical system.[1] For the development of theory, the noise is usually assumed to be Gaussian distributed and may either be of infinite bandwidth (delta-function correlated), so called "white noise" or of finite bandwidth (and hence correlation time), so called "colored noise". The noise actually encountered in physical and natural applications is, however, always colored and one is fortunate if it is even approximately Gaussian. We are interested here in the unusual effects that noise has upon dynamical systems when it is a part of the dynamics. This in contrast to "instrumental noise" or that which is simply added to the output of some system.[2] The former accounts for a wide variety of new dynamical processes - new in the sense that these processes would not emerge in the absence of the noise - while the latter cannot alter the qualitative behavior of the system and only "masks" the responses that cleanly appear in the absence of noise. What are some of these new dynamical processes? Examples of some of the original processes are: the statistical postponements of bifurcations,[3] noise induced transitions,[4] stochastic resonance[5] or changes in the topology of the probability densities of response functions,[6] the statistical selection of outcomes in complex systems,[7] and "hole burning" in the statistical density by the noise correlation time.[8] The study of these processes opened the door to more recent discoveries including stochastic resonance in laser[9] and biological systems,[10] the development and application of a related process, synchronization in chaotic and noisy systems,[11] coherence resonance,[12] the generalization of the noise-induced phase transition,[13] and the discovery of d oubly stochastic resonance.[14] In the ensuing sections we shall touch these in more detail.

The aforementioned stochastic processes are characteristic of nonlinear systems in particular. And the most extreme form of nonlinearity is the so-called "hard" threshold. Stochastic resonance, for example, is most easily pictured as consisting of three ingredients: a threshold, a subthreshold, information-carrying signal and noise. The paradigm is that no information can flow through the system unless the threshold is crossed and it cannot be crossed by the signal alone. The addition of noise to the signal thus results in a sequence of threshold crossings that contain information about the subthreshold signal. Just the right amount of noise - the optimal noise intensity - maximizes the information content in the train of threshold crossings. This somewhat counterintuitive phenomenon is illustrated in Fig.1 where we show the threshold (dashed line) and a randomized subthreshold signal (solid line) and the added noise (irregular line). The threshold crossings are depicted above by vertical markers. They can be encoded as a binary sequence of “0” (absence of corssing) and “1” (threshold crossing). The transinformation[16] between a set of threshold crossings in the absence of the signal and those with

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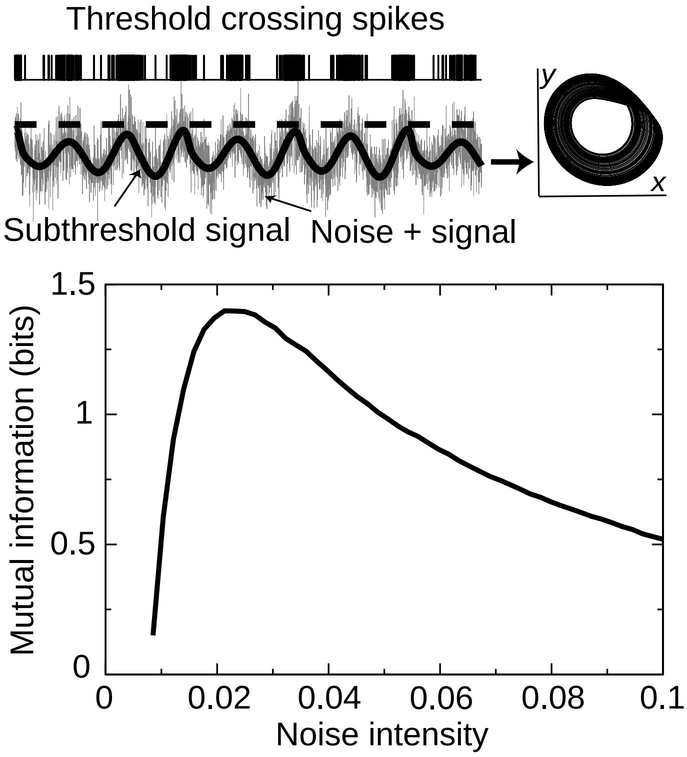

Figure 1. Noise-enhanced information transfer in a threshold system. The subthreshold signal $x(t)$ was generated by the Rössler system: $\dot{x} = -(y + z)$, $\dot{y} = x + 0.15y$, $\dot{z} = 0.2 + z(x - 7.1)$. (see inset on the right showing $x - y$ projection of the Rössler attractor). Colored Gaussian noise $\xi(t)$ was generated by the following stochastic differential equation: $\ddot{\xi} + 2/\tau \dot{x}i + 1/\tau\xi = \sqrt{2D}/\tau^2 w(t)$, where $\tau = 0.05$ is the noise correlation time, D is the noise intensity and $w(t)$ stands for white Gaussian noise.

the signal present is shown below on the right. The transinfromation, I was estimated as a difference between the Shannon entropy calculated for the thrshold crossing sequence without signal, H_0 and one with signal H_s: $I = H_0 - H_s$. For this demonstration we have used an aperiodic subthreshold signal generated by the chaotic Rössler system.

This behavior shown in lower panel of Fig.1 – the maximization of some information or signal quality measure at an optimal noise intensity – is called stochastic resonance (SR), and a huge literature on this subject exists. We point here only to two reviews: a quite recent one[15] and a classical one.[17] In the sequel we will discuss some of the classical and recent experiments on SR. Further we will outline the modern development and applications of stochastic phase synchronization (SS) and discuss the relation between the two processes.

However, as a prelude to this discussion, we will outline some older processes and a few of the experiments they inspired. These generally are qualitatively new dynamical processes that are induced by noise. They could, therefore fall under the heading of "Why Noise in Interesting", a theme that recurs throughout this essay.

2. WHY NOISE INTERESTING

Actually it is not interesting at all in linear systems, serving only to obscure the information content of useful signals, as all engineers know well. But in nonlinear systems a variety of interesting and challenging processes occur. These occur when the noise is dynamical, that is, it is an inherent part of the dynamics of the system. Most such systems are described by a Langevin equation:

$$\dot{\mathbf{x}} = \mathbf{F}(\boldsymbol{x}) + \mathbf{G}\,(\boldsymbol{x})\boldsymbol{\xi}(\boldsymbol{t}), \tag{1}$$

where $\mathbf{x}(t)$ is a state variable in N-dimensional space, $\mathbf{F}(t)$ is the deterministic part of the evolution operator, while the noise sources $\xi(t)$ together with a nonlinear function $\mathbf{G}(x)$ corresponds to stochastic perturbations. In the case when noise sources are Gaussian and white (e.g. delta correlated) the stochastic process $\mathbf{x}(t)$ is Markovian and its properties can be studied in terms of the Fokker-Planck equation.[18]

As an example we present stochastic Duffing oscillator:

$$\ddot{x} + \gamma\dot{x} = x - x^3 + \xi(t), \tag{2}$$

where $\xi(t)$ is the noise and γ is the friction coefficient. The noise drives the system point randomly over a barrier separating the two wells of the "standard quartic" potential,

$$U(x) = -\frac{x^2}{2} + \frac{x^4}{4}. \tag{3}$$

The solution is the probability density, P(x,t) given by the Fokker-Planck equation[18] which cannot be solved exactly except in the limit of large friction,

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{\gamma}\left[-\frac{\partial}{\partial x}F(x) + D\frac{\partial^2}{\partial x^2}\right]P(x,t), \tag{4}$$

where $F(x) = x - x^3$ is the forcing in Eq.(2), and D is the diffusion coefficient, or noise intensity.

There are, of course other scenarios that differ in details from the foregoing outline, for example in the form of $F(x)$ or $\xi(t)$. Indeed, the one-dimensional, one-particle system can be generalized to many particles diffusing and forming patterns in many spatial dimensions. We will specify these differences as needed below.

2.1. Postponed bifurcations

An early example of the qualitatively different behavior of dynamical noise was illustrated by the postponement of bifurcations. The first instance[3] occurred in a system similar to that specified by Eq.(2)except the noise is now multiplicative on an additional quadratic term in the forcing, $\zeta(t)x^2$, with $\zeta(t) \to \langle\zeta\rangle + \xi(t)$. The bifurcation was from noisy fixed point to bistable behavior[20] with $\langle\zeta\rangle$ as the control parameter. Postponement means that the critical value, $\langle\zeta\rangle_{crit1}$ was pushed to larger values by increasing intensity of the noise, $\xi(t)$. Finally at a second value $\langle\zeta\rangle_{crit2}$ bistability was destroyed. This second transition was evident in a change in topology of the stationary probability density, $P(x)$, from having three extrema (two maxima, one minimum) to one (a single maximum).

But there are also postponements in oscillatory systems. A classic example is the noise mediated Andronov–Hopf bifurcation in, for example, the Brusselator[21] defines by,

$$\begin{aligned} \dot{x} &= A - (1+B)x + x^2 y, \\ \dot{y} &= Bx - x^2 y \end{aligned} \tag{5}$$

with the noisy bifurcation parameter given by $B(t) \to \langle B\rangle + \xi(t)$. We define the presence or absence of the limit cycle by the topology of its 2-D stationary probability density $P(x,y)$. If $P(x,y)$ has any cross section (parallel to the xy plane) that is doubly connected, then we say that the limit cycle exists. An example is shown in Fig.2. The "crater" is the central depression. "Crater Lake" holds water when a cross section is doubly connected as shown in Fig.2 (left panel) for $\langle B\rangle > \langle B\rangle_{crit}$. The water just does drain when the cross section becomes simply connected for which $\langle B\rangle = \langle B\rangle_{crit}$, that is, $\langle B\rangle_{crit}$ is the largest value of the bifurcation parameter for which all cross sections are simply connected. Here, the noise was exponentially correlated,

$$\langle\xi(t)\xi(s)\rangle = \frac{D}{\tau}\exp\left(-\frac{|t-s|}{\tau}\right), \tag{6}$$

with a fixed correlation time, τ. Postponement means that $\langle B\rangle_{crit}$ increases with increasing noise intensity, D.

Other topological changes in the stationary density mediated by the noise correlation time have been observed.[8,22] So called "holes" in the cross sections (transitions from singly to doubly connected topology) of $P(x,y)$ of an overdamped (large γ) Duffing system, Eq.(2) were induced solely by increasing the noise correlation time.

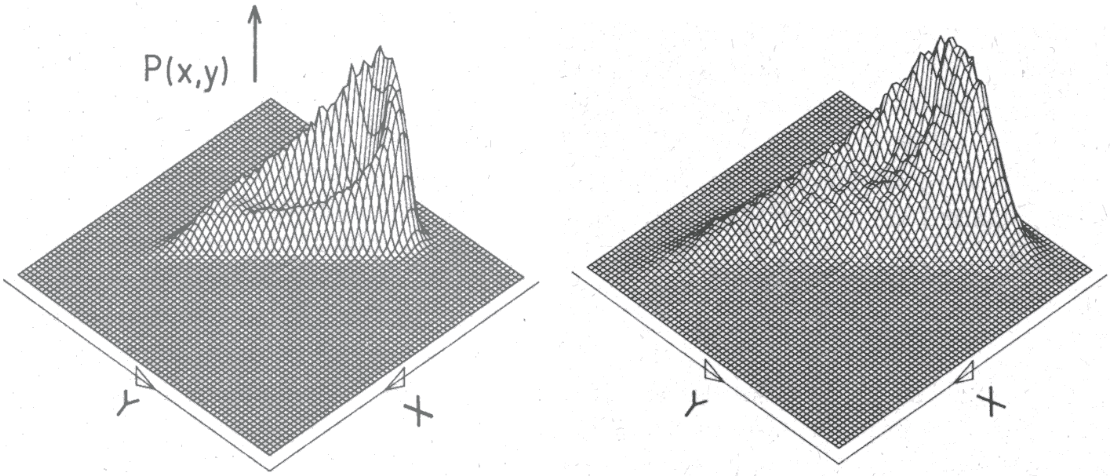

Figure 2. The noisy limit cycle of the Brusselator (left) for $\langle B\rangle > \langle B\rangle_{crit}$ and an example of $P(x, y)$ for $\langle B\rangle = \langle B\rangle_{crit}$.

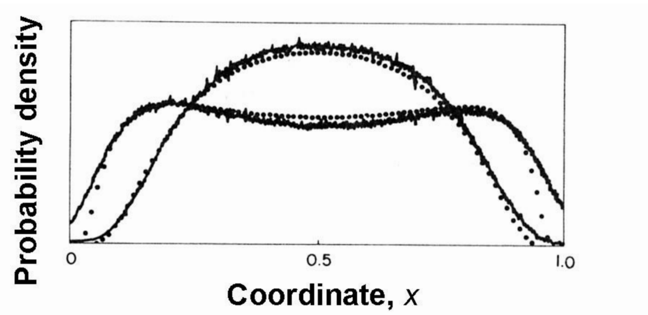

Figure 3. Noise induced change in topology of the stationary probability density of the genetic model, Eq.(7). The continuous curves are measured with the analog simulator, the dots are calculated from the solution of a Fokker-Planck equation corresponding to Eq.(7). $D_{crit} = 2.0$ (arbitrary units) for this system. Curve with single maximum, $D = 1.5 < D_{crit}$. Curve with double maxima, $D = 2.5 > D_{crit}$.

2.2. Noise–induced transitions

Following Horsthmeke and Lefever[23] we define a noise-induced transition as a topological change in the stationary probability density that happens at some critical noise intensity. Noise-induced transitions have been observed in a wide variety of systems. We mention here only a few. Foreshadowing much now contemporary research, the paradigmatic system was biologically motivated.[23] An elementary genetic model is given by,

$$\dot{x} = \frac{1}{2} - x + \zeta(t)x(1 - x), \tag{7}$$

where again we are suing multiplicative noise with a nonzero mean: $\zeta(t) \to \langle\zeta\rangle + \xi(t)$. The forcing in this system – the right-hand-side of Eq.(7) – is a kind of "soft threshold". It is "S" shaped, and depending on the value of $\langle\zeta\rangle$ can be either bistable or monostable. We choose to operate the system as a deterministic monostable device. Then bistability, in the statistical sense of the appearance of two peaks in the stationary probability density $P(x)$, can be induced solely by the noise intensity D. When $D > D_{crit}$ the probability density has three extrema, but when $D \leq D_{crit}$ there is only one. An example is shown in Fig.3.

An order parameter for the transition can be defined as the distance separating the two peaks in the bistable regime. This transition was observed and analyzed by analog simulation.[24] Other similar noise-induced topological transitions were observed in a model for turbulent superfluid helium[25] and in the general bistable system

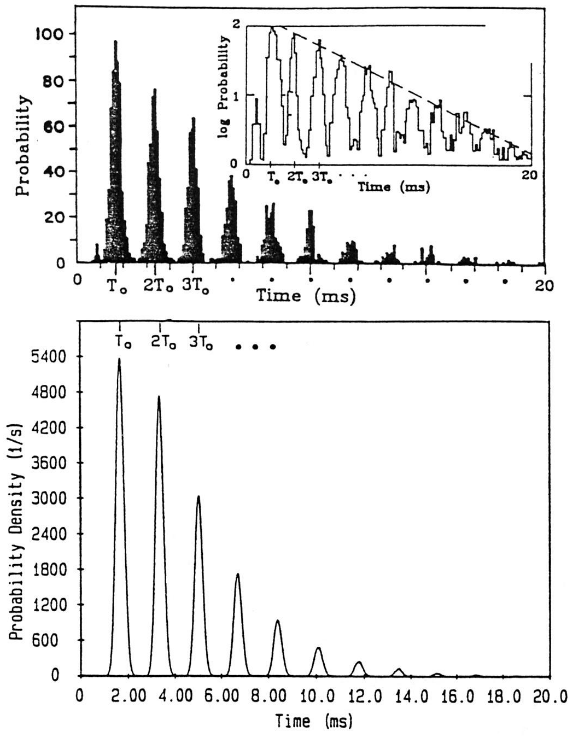

Figure 4. Upper panel: Interspike interval histograms (ISIH) of responses of auditory neuron subjected to weak stimulus with period 1.66 ms from reference.[25] Inset: data plotted on semi logarithmic scale. Lower panel: ISIH from an experiment with an electronic Schmitt trigger from reference.[26] In both cases the peaks are located at integer multiples of the stimulus period.

in conjunction with postponements.[20] Studies of these transitions have recently been generalized to represent true phase transitions[13] (rather than simply topological transitions in a stationary density) and in conjunction with stochastic resonance to produce *doubly stochastic resonance*.[14]

2.3. Thresholds and noise in neural coding

A phase locking phenomenon well known in sensory biology, called skipping, occurs when a periodic stimulus is applied to some sensory neuron at the subthreshold level, for example a weak tone to an auditory system. Then the neuron does not fire except when its membrane threshold potential is crossed. This can only happen when noise is added to the stimulus. The noise is usually internal to the neuron. It is most probable that the threshold will be crossed (thus generating an action potential) when the maxima of the stimulus approach the membrane potential. It is less probable that a stimulus cycle will be "skipped" and (exponentially) even less probable that two cycles will be skipped and so on. This results in post stimulus time histograms of the time intervals between neural action potentials with peaks located at integer multiples of the stimulus period. Moreover the amplitudes of the peaks are an exponentially decreasing function of the number of cycles skipped. An example is shown in Fig.4. The upper panel shows data from squirrel monkey auditory system[25] with the inset showing the same data plotted on a semi logarithmic scale. The straight dashed line indicates an exponential decrease in successive peak amplitudes.

The lower panel reports the results of an experiment with an electronic Schmitt trigger, the simplest possible threshold device.[26] The trigger was subject to a subthreshold periodic signal (period 1.66 ms, the same as for the squirrel monkey experiment) plus Gaussian band limited noise. The ISIH from the trigger demonstrates the principle underlying the widely observed phenomenon called "skipping". It is significant to note that the multiple, exponentially decaying peaks at integer multiples of the stimulus period cannot occur in the absence of noise. Evidence exists that animals do use the phase locking or skipping phenomena to interpret weak sounds. Thus internal neural noise, in this instance, is essential for coding and information transmission. These results were highlighted in an editorial commentary published in Nature by John Maddox.[27]

In the subsequent Section we will discuss the modern manifestation of this phase locking phenomenon and demonstrate its foundation in statistical physics.

2.4. Synchronization of noisy systems

The fundamental phenomenon of synchronization occurs if a periodic self-sustained oscillator (that is, an oscillator possessing a stable limit cycle in its phase space) is driven by a periodic force or coupled to another self-sustained oscillator. From its etymology synchronization means adjustment of rhythms of interacting oscillators.[11] Synchronization can be defined as phase locking of oscillators, when at certain parameter values (for example, coupling strength and frequency ratio of two oscillators) oscillations in two systems occurs in concert (or in synchrony) so that the phase difference does not change in time. Otherwise, synchronization is defined as frequency entrainment, when changing the frequency on one oscillator (or periodic force) leads to correspondent adjustment of the frequency of another oscillator. If we denote the phase of one oscillator as $\Phi(t)$ and the phase of another oscillator (or periodic force) by $\Psi(t)$ then the condition for synchronization is $|n\Phi(t) - m\Psi(t)| < const$ in terms of phase locking or $n\langle\dot{\Phi}(t)\rangle = m\langle\dot{\Psi}(t)\rangle$, in terms of frequency entrainment, where m and n are intergers in both cases. The important moment here is that these condition should be fullfilled not just for a few parameters values, but rather in a region in the parameter space. These regions are called "synchronization regions" or Arnold tongues.

Phase synchronization was recently generalized to the case when interacting oscillators have complicated chaotic dynamics instead of just being periodic.[28] In such a case intantaneous phase can be introduced via various approaches, for example using the analytic signal approach and the Hilbert transform.[11] The essence of this chaotic phase synchronization is that the phases of oscillators can be locked just as in the case of periodic oscillators, while the instantaneous amplitude of oscillators are still weakly correlated and chaotic.

Indeed synchronization has found huge numbers of application in engineering and is a significant mechanism of dynamical operation in biological systems. But all real oscillatory systems, epsecially ones in biology are inevitably contaminated by noise. The effect of noise on a periodic self-sustained oscillators are well-known due to Stratonovich[29]: noise leads to fluctuations of the amplitude and to diffusion of the phase of oscillator and if noisy oscillators are coupled than their phase difference diffusses as well. Thus, in a usual situation noise always plays against synchronization. In the presence of noise the very notion of synchronization becomes "blurred" and it has to be considered in a statistical sense by introducing some restrictions to statistical measures of underling stochastic processes. In this way we arrive to the notion of "effective"[30] or "stochastic"[11,31,32] synchronization. It can be defined by imposing restrictions on

1 signal-to-noise ratio, in the case of periodically driven self-sustained oscillator;

2 frequency fluctuations;

3 phase fluctuations.

In the case of simple 1:1 synchronization ($m = n = 1$) the dynamics of the phase difference, $\phi(t)$ of two coupled oscillators or one oscillator driven by periodic force, can be approximated by the Adler equation[33]:

$$\dot{\phi} = \Delta - \epsilon \sin\phi + \sqrt{2D}\,\xi(t), \tag{8}$$

where Δ is the frequency mismatch and the parameter ϵ measures the strength of coupling. In the absence of noise ($D \to 0$) the synchronization regime occurs when $\Delta < \epsilon$ and refers to the existence of a stable equilibrium, $\phi_s = \arcsin(\Delta/\epsilon)$. The stable equilibium is accompanied by unstable one at $\phi_u = \arcsin(\Delta/\epsilon + \pi)$. With noise taken into account the phase difference performs the Brownian motion in a periodic tilted potential $U(\phi) = -\Delta\phi - \epsilon\cos(\phi)$. The minima of the potential $U(\phi)$ corresponds to the stable equilibria ϕ_s, while the potential maxima located at the unstable equilibria ϕ_u. If $\Delta < \epsilon$, then for weak noise the phase difference "Brownian particle" will diffuse over the potential profile in the following way: it will perform a small-scale fluctuations inside potential wells and rarely jump from one potential well to another. These jumps from one potential well to another are called "phase slips" whereby the phase difference changes ubruptly by 2π. The long stay of the phase difference (although with small-scale fluctuations) inside a potential well corresponds to the phase locking behavior. The longer are these phase locking segments, the stronger is synchronization.

The statistical measures of synchronization can be based on the stationary probability density of the phase difference wrapped into $[0\,2\pi]$. A well-expressed maximum will corresponds to a strong synchronization in statistical sense. This can be further quantified by the synchronization index[11] as a first Fourier mode of the stationary probability density of the phase difference:

$$\gamma^2 = \langle \sin\phi \rangle^2 + \langle \cos\phi \rangle^2. \tag{9}$$

The synchronization index changes from 0 (no synchronization, unfirom distribution of the phase difference) to 1 (perfect synchronization, δ-type distribution of the phase difference).

Another way to characterize stochastic synchronization is to calculate the effective diffusion coefficient. Slow diffusion over the periodic tilted potential $U(\phi)$ corresponds to the long phase locking segments. Thus, small in comparison with fundamental oscillator periods effective diffusion coefficient is another way to characterize stochastic synchronization.[29]

As we already noted above, phase synchronization is conventional oscillatory system, described by Eq.(8) becomes worse as noise increases: the synchronization coefficient decreases and the effective diffusion coefficient increases with the increase of noise intensity D.[11, 29] Surprisingly, in systems exhibiting stochastic resonance situation can be quite opposite. Periodically driven bistable or excitable stochastic systems can be considered from the synchronization point of view.[32, 34] For example, the phase of a stochastic bistable system can be associated with the moments of time when a particle crosses the barrier height[32] or simply with a level crossing.[35] Then we can pose synchronization problem: whether the instantaneous phase of the switching and the corresponding mean switching frequency can be locked by external periodic force. In[32, 36] it was shown that stochastic resonance actually can be considered as noise-enhanced phase locking phenomenon. The mean switching frequency can be locked in a finite range of noise intensities, while the effective diffusion coefficient excibits a minimum being plotted versus the noise intensity. In[31] mutual synchronization of two coupled stochastic bistable systems was studied. An analytical approach for calculations of the effective diffusion constant was developed in.[37] Thus, the notion of synchronization can be extended to a wide class of systems whose characteristic time scales are completely controlled by noise.

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