Quantum States, Kets, and State Space

The formalism of quantum mechanics involves symbols and methods for denoting and determining the time-dependent **state** of a physical system, and a mathematical structure for evaluating the possible outcomes and associated probabilities of measurements that can be made on the system. Here, "state" implies everything knowable about the dynamical aspects of a system at a certain time.

In **classical mechanics**, the state at time t of a particle of mass m is given by the particle's position $\mathbf{r}(t)$ and momentum $\mathbf{p}(t)$, which are determined by initial conditions and the laws of classical mechanics. Neither m nor t are state variables: m is assumed to be an immutable property of the particle, and t is a parameter used in specifying the dynamics of state variables.

In **quantum mechanics**, the state of a physical system at time t is associated with a symbol such as ψ placed within half-right-angled brackets: $|\psi\rangle$. This symbol is called a **ket vector** or a **ket**. A symbol within a ket may represent a physical quantity. For instance, if E_1 is a possible energy of a particle, then $|E_1\rangle$ indicates that the particle is in a state with an energy of precisely E_1. Kets may contain multiple symbols that represent multiple physical quantities.

A ket is the quantum mechanical symbol that encodes the state of a system. The symbols within a ket explicitly indicate or serve as an abstract placeholder for precisely known information about the system.

The mathematical structure of quantum mechanics formalism mirrors that of **linear algebra** (see pages 113–114) and encapsulates rules regarding the manipulation of kets to determine the time dependence of a system's state and how statistical information (such as measurement outcome predictions) about a system may be determined from its state. The **state space** $\mathcal{E}$ for a given physical system is the set of all of the possible states in which it can exist. The statement $|\psi\rangle \in \mathcal{E}$ indicates that $|\psi\rangle$ encodes a state in which the given system can exist.

Elements of Dirac Notation

Dirac notation is a standard system of notation used in quantum mechanics that efficiently keeps track of information about states. Expressions given in Dirac notation can often be replaced by expressions involving the elements and operations of calculus or linear algebra. However, the system of Dirac notation is compact, adaptable to any quantum mechanics problem, and can be related to other mathematical and notational systems as needed. Dirac notation involves four categories of elements: kets (page 1), complex scalars, operators, and bras.

Operators: In this *Field Guide*, operators are denoted in Dirac notation with a "hat" or caret placed over a letter or symbol, as in $\hat{X}$. Outside of Dirac notation, operators may be represented by matrices, or by operations such as multiplication or differentiation; in these cases the caret is not used in this *Field Guide*. Most operators in this *Field Guide* are denoted with capital letters, with a few exceptions (see the Glossary of Operators, pages xvi–xvii). Formally, an operator is a quantity that associates every ket in a given state space $\mathcal{E}$ with the same or another ket. For an arbitrary operator $\hat{A}$ and an arbitrary ket $|\psi\rangle \in \mathcal{E}$, this association is denoted $\hat{A}|\psi\rangle = |\psi'\rangle$. Operators are therefore said to "act to the right" on kets to produce new kets.

Many operators are associated with measurable physical quantities. Most operators of mathematical utility that do not correspond to physical quantities are written in this *Field Guide* in a "hollow" typeface, such as $\hat{\mathbb{U}}$ and $\hat{\mathbb{P}}$. All projection operators and unitary operators (except the Pauli spin operators) are written in this typeface throughout this *Field Guide*.

Bras: A bra or **bra vector** is a symbol placed within a half-left-angled bracket, such as $\langle\varphi|$. A bra is a **functional**: it associates a complex scalar with each ket in a given state space, whereas operators associate a ket with another ket. A bra placed immediately to the left of a ket creates a bra–ket compound symbol such as $\langle\varphi|\psi\rangle$ that is equivalent to a complex scalar. As with kets, the symbol(s) inside a bra designates information about the properties of the bra. Operators may "act to the left" on bras to produce new bras, as indicated by the expression $\langle\varphi|\hat{A} = \langle\varphi'|$.

Vector Spaces and Scalar Products

The state space $\mathcal{E}$ of an arbitrary quantum-mechanical system has the properties of a **vector space**. The dimensionless elements of $\mathcal{E}$ symbolized by kets are called the **state vectors** of $\mathcal{E}$. The principle of **superposition** enables the mathematical construction of new elements of $\mathcal{E}$ from other elements of $\mathcal{E}$; e.g., if $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are elements of $\mathcal{E}$, then

$$c_1|\varphi_1\rangle + c_2|\varphi_2\rangle \equiv |\varphi'\rangle \in \mathcal{E}$$

where c_1 and c_2 are complex scalars.

The **scalar** or **inner product** of two elements of $\mathcal{E}$ is denoted by a bra–ket pair. The scalar product associated with the ordered pair $(|\varphi\rangle, |\psi\rangle)$ is written with the second ket preceded by a bra containing the symbol(s) within the first ket; i.e., $\langle\varphi|\psi\rangle$. The bra $\langle\varphi|$ is called the **adjoint** (or **Hermitian conjugate**) of $|\varphi\rangle$, and symbolizes the concept that for every ket, there is a corresponding bra. State vectors and their scalar products have the following properties:

1. Scalar products are generally complex. Reversing the order of the kets in the ordered pair $(|\varphi\rangle, |\psi\rangle)$ is equivalent to complex conjugation. This property is written in bra–ket notation as $\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$

2. **Anti-linearity** in the 1^{st} term: if $|\varphi'\rangle \equiv c_1|\varphi_1\rangle + c_2|\varphi_2\rangle$, then

$$\langle\varphi'|\psi\rangle = c_1^*\langle\varphi_1|\psi\rangle + c_2^*\langle\varphi_2|\psi\rangle$$

3. **Linearity** in the 2^{nd} term: if $|\psi'\rangle \equiv d_1|\psi_1\rangle + d_2|\psi_2\rangle$, then

$$\langle\varphi|\psi'\rangle = d_1\langle\varphi|\psi_1\rangle + d_2\langle\varphi|\psi_2\rangle$$

4. Any ket $|\psi\rangle$ that corresponds to a physically realizable quantum state must be **normalizable**, meaning that the **norm**

$$\||\psi\rangle\| \equiv \sqrt{\langle\psi|\psi\rangle}$$

is real, finite, and positive. $|\psi\rangle$ is normalized if $\||\psi\rangle\| = 1$. If $\||\psi\rangle\| \neq 1$, then dividing $|\psi\rangle$ by $\sqrt{\langle\psi|\psi\rangle}$ normalizes $|\psi\rangle$.

5. If the scalar product of two normalizable state vectors is zero, then the state vectors are said to be **orthogonal**.

Linear Operators and Commutators

The formalism of quantum mechanics involves **linear** operators. Consider two operators $\hat{A}$ and $\hat{B}$ that act on elements of an arbitrary state space $\mathcal{E}$. Also suppose the following: (i) $|\psi\rangle, |\varphi\rangle \in \mathcal{E}$; (ii) $\hat{A}|\psi\rangle = |\psi'\rangle \in \mathcal{E}$; (iii) $\hat{B}|\psi\rangle = |\psi''\rangle \in \mathcal{E}$; (iv) c_1 and c_2 are complex scalars. **Linearity** implies the following:

1. $\hat{A}(c_1|\psi\rangle + c_2|\varphi\rangle) = c_1\hat{A}|\psi\rangle + c_2\hat{A}|\varphi\rangle$
2. $\hat{B}\hat{A}\,|\psi\rangle = \hat{B}(\hat{A}|\psi\rangle) = \hat{B}|\psi'\rangle$
3. $\hat{A}\hat{B}\,|\psi\rangle = \hat{A}(\hat{B}|\psi\rangle) = \hat{A}|\psi''\rangle$

The outcomes of the sequential operations $\hat{A}\hat{B}$ and $\hat{B}\hat{A}$ are generally different. When products of operators act on a ket, a standard **order of operations** is used: the rightmost operator acts first on the ket, then the next operator to the left acts on the new ket that is the outcome of the first operator's action, and so on.

Commutators: The commutator of operators $\hat{A}$ and $\hat{B}$ is denoted and defined as

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Commutators have the following properties:

1. A commutator is itself an operator. If $\hat{C} \equiv [\hat{A}, \hat{B}]$, then

$$\hat{C}|\psi\rangle = \hat{A}\hat{B}\,|\psi\rangle - \hat{B}\hat{A}\,|\psi\rangle = \hat{A}|\psi''\rangle - \hat{B}|\psi'\rangle$$

2. If $[\hat{A}, \hat{B}] = 0$, then $\hat{A}$ and $\hat{B}$ **commute**: the order in which they act on any ket does not matter, and $\hat{A}|\psi''\rangle = \hat{B}|\psi'\rangle$.

3. If $[\hat{A}, \hat{B}] \neq 0$, then the order of operations of $\hat{A}$ and $\hat{B}$ is **noncommutative**: the order of operations does matter. For arbitrary $|\psi\rangle$, if $[\hat{A}, \hat{B}] \neq 0$, then

$$\hat{A}\hat{B}\,|\psi\rangle \neq \hat{B}\hat{A}\,|\psi\rangle$$

4. A **commutation relation** for operators $\hat{A}$ and $\hat{B}$ identifies the specific operator that is equivalent to $[\hat{A}, \hat{B}]$, which may be identical to zero, or to a scalar multiple of the identity operator $\hat{\mathbb{I}}$ (in which case $\hat{\mathbb{I}}$ is often omitted from the commutation relation).

Hermitian Conjugation

Hermitian conjugation involves symbolic manipulation of mathematical expressions. Consider elements $|\psi\rangle$ and $|\varphi\rangle$ of a state space $\mathcal{E}$, and a bra $\langle\chi|$ and operator $\hat{A}$ defined to "act on" elements of $\mathcal{E}$. Let $\hat{A}|\psi\rangle = |\psi'\rangle$. Hermitian conjugation involves the following concepts and rules:

1. **Bra–ket correspondence:** for every ket $|\psi\rangle$, there is a corresponding bra $\langle\psi|$. For every bra $\langle\chi|$, there is a corresponding ket $|\chi\rangle$ that is not necessarily an element of $\mathcal{E}$. $|\psi\rangle$ and $\langle\psi|$ are Hermitian conjugates of one another, as are $|\chi\rangle$ and $\langle\chi|$.

2. The **Hermitian conjugate**, or **adjoint**, of $\hat{A}$ is denoted $\hat{A}^\dagger$. If $\hat{A} = \hat{A}^\dagger$, then $\hat{A}$ is said to be **Hermitian**.

3. An operator can act "to the right" on a ket, or "to the left" on a bra. The quantity $\langle\varphi|\hat{A}|\psi\rangle$ is called a **matrix element** of $\hat{A}$ and can be evaluated as $\langle\varphi|\hat{A}|\psi\rangle = \langle\varphi|(\hat{A}|\psi\rangle) = \langle\varphi|\psi'\rangle$, or equivalently as $\langle\varphi|\hat{A}|\psi\rangle = (\langle\varphi|\hat{A})|\psi\rangle = \langle\varphi'|\psi\rangle$, where $\langle\varphi'|$ is the adjoint of $|\varphi'\rangle = \hat{A}^\dagger|\varphi\rangle$ (see item 6 below).

4. $\hat{A}$ acting (to the left) on $\langle\xi| = \lambda_1\langle\psi| + \lambda_2\langle\varphi|$ is linear:

$$\langle\xi|\hat{A} = \lambda_1\langle\psi|\hat{A} + \lambda_2\langle\varphi|\hat{A}$$

where λ_1 and λ_2 are complex scalars.

5. Hermitian conjugation of operators and operator sequences adheres to the following rules, where $\hat{A}$, $\hat{B}$, and the ket–bra pairs $|\varphi\rangle\langle\psi|$ and $|\psi\rangle\langle\varphi|$ are operators, and λ is a scalar:

$$(\hat{A}^\dagger)^\dagger = \hat{A}, \qquad (\lambda\hat{A})^\dagger = \lambda^*\hat{A}^\dagger$$

$$(|\varphi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\varphi|, \qquad (\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$$

$$(\hat{A}+\hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger = \hat{B}^\dagger + \hat{A}^\dagger = (\hat{B}+\hat{A})^\dagger$$

6. To obtain the Hermitian conjugate of an expression, first take the complex conjugates of all scalars, replace kets with their corresponding bras, replace bras with their corresponding kets, and replace operators with their adjoints; then reverse the order of all elements, noting that scalars commute with all elements. For example, the Hermitian conjugate of the expression $\hat{A}|\psi\rangle = \lambda|\psi'\rangle$ is $\langle\psi'|\lambda^* = \langle\psi|\hat{A}^\dagger$.

Bases

A **basis** for $\mathcal{E}$ is a set of kets that are often (but not necessarily) elements of $\mathcal{E}$, and which can be used to construct any element of $\mathcal{E}$ via **superposition**. The elements of a basis may be **discretely** or **continuously** indexed, depending on the basis. For the arbitrary **discrete basis** $\{|v_k\rangle\}$, the **index** k numbers the elements of the basis; e.g., $\{|E_1\rangle, |E_2\rangle, \ldots\}$ might be the notation used for a basis where the labels indicate possible energies of a particle in a potential well. For a **continuous basis** $\{|w_\beta\rangle\}$, the subscript β is a continuously varying index, such as real numbers that might correspond to positions along a coordinate axis. The bases used in this *Field Guide* have the following properties:

1. For every $|\psi\rangle \in \mathcal{E}$, there is one and only one way to expand $|\psi\rangle$ as a superposition of the elements of any particular basis, up to a **global phase** factor (page 36). The superposition is written

$$|\psi\rangle = \sum_k c_k |v_k\rangle \quad \text{(for a discrete basis)}$$

$$|\psi\rangle = \int_{\text{all } \beta} d\beta\, c(\beta) |w_\beta\rangle \quad \text{(for a continuous basis)}$$

There is therefore one unique set of **expansion coefficients** $\{c_k\}$ (discrete basis) or $\{c(\beta)\}$ (continuous basis) for any state vector's expansion into any given basis, where $c_k = \langle v_k|\psi\rangle$ for any k, and $c(\beta) = \langle w_\beta|\psi\rangle$ for any β. The sets $\{|v_k\rangle\}$ and $\{|w_\beta\rangle\}$ are said to **span** the state space $\mathcal{E}$ if they are bases for $\mathcal{E}$.

2. Every discrete basis used in this *Field Guide* is **orthonormal** (all elements are normalized and mutually orthogonal). This means that for a basis $\{|v_k\rangle\}$, $\langle v_j|v_k\rangle = \delta_{jk}$, where δ_{jk} is the **Kronecker delta** (page 112).

3. The continuous basis $\{|w_\beta\rangle\}$ is conventionally said to be orthonormal if $\langle w_{\beta'}|w_\beta\rangle = \delta(\beta' - \beta)$, where $\delta(\beta' - \beta)$ is the **Dirac delta function** (page 112). Although this condition is not normalization in the strictest sense, the elements of a continuous basis are not physically realizable quantum states and do not belong to the state space $\mathcal{E}$, despite forming a basis for $\mathcal{E}$.

Eigenvalue Equations

An **eigenvalue equation** for an arbitrary operator $\hat{A}$ is similar to that of a matrix (page 114), and takes the form

$$\hat{A}|\varphi_\mu\rangle = \lambda_\mu|\varphi_\mu\rangle$$

where μ is a discretely or continuously varying index. For a given operator $\hat{A}$, the equation is satisfied for a particular set of kets $\{|\varphi_\mu\rangle\}$, where each ket $|\varphi_\mu\rangle$ is associated with a scalar λ_μ (generally complex). The kets of the set $\{|\varphi_\mu\rangle\}$ are the **eigenkets** or **eigenstates** of $\hat{A}$. The scalars of the set $\{\lambda_\mu\}$ are the **eigenvalues** of $\hat{A}$. The set $\{\lambda_\mu\}$ is the eigenvalue **spectrum** of $\hat{A}$. Eigenvalue equations have the following characteristics:

1. The eigenkets are indexed, as indicated above by μ. Often the symbol that labels an eigenket is the associated eigenvalue or the index. The eigenvalue equation above and the two below show three equivalent ways of labeling the eigenkets of $\hat{A}$:

$$\hat{A}|\lambda_\mu\rangle = \lambda_\mu|\lambda_\mu\rangle \quad \text{and} \quad \hat{A}|\mu\rangle = \lambda_\mu|\mu\rangle$$

2. Quantum mechanics problems often involve finding the eigenkets and the eigenvalues of operators. It is possible that an operator does not have solutions to its eigenvalue equation.

3. If a basis for a state space $\mathcal{E}$ has n orthonormal elements, then $\mathcal{E}$ is said to have a **state-space dimension** of n, to be distinguished from a coordinate-space dimension. For any Hermitian operator that acts on elements of an n-dimensional state space $\mathcal{E}$, a set of n distinct mutually orthonormal eigenkets of that operator can be found; this set is a basis that spans $\mathcal{E}$.

4. **Degeneracy** means that $g_\mu > 1$ different orthogonal eigenkets of an operator are associated with the same eigenvalue λ_μ; g_μ is the **degree of degeneracy**. An additional index may label these degenerate eigenkets; e.g., for a given eigenvalue λ_μ,

$$\hat{A}|\varphi_\mu^i\rangle = \lambda_\mu|\varphi_\mu^i\rangle; \quad i \in \{1, 2, \cdots g_\mu\}$$

The superscript i is omitted from $|\varphi_\mu^i\rangle$ if $g_\mu = 1$ for a given λ_μ. If $g_\mu > 1$ for one or more values of λ_μ, then various sets of mutually orthogonal eigenkets can be specified. If no degeneracies exist, then there is one unique set of orthonormal eigenkets (up to global phase factors; see page 36).

Projectors

A **projector** or **projection operator** onto $|\psi\rangle$ is defined as $\hat{\mathbb{P}}_\psi \equiv |\psi\rangle\langle\psi|$. Projectors are operators, as seen in the following:

$$\hat{\mathbb{P}}_\psi |\varphi\rangle = (|\psi\rangle\langle\psi|)\, |\varphi\rangle = |\psi\rangle \langle\psi|\varphi\rangle = \lambda\, |\psi\rangle$$

where $\lambda = \langle\psi|\varphi\rangle$ is a complex scalar, so that $\hat{\mathbb{P}}_\psi$ produces a new (and generally unnormalized) ket proportional to $|\psi\rangle$.

If the scalar product $\lambda = \langle\psi|\varphi\rangle$ is non-zero, then $|\varphi\rangle$ and $|\psi\rangle$ are said to **overlap**: $\langle\psi|\varphi\rangle$ quantifies how much the state $|\psi\rangle$ and $|\varphi\rangle$ have in common (and thus "overlap") in terms of their expansions into the same basis. If the states have non-zero overlap, then there are basis elements that appear with nonzero coefficients in the expansions of both $|\psi\rangle$ and $|\varphi\rangle$. $\hat{\mathbb{P}}_\psi|\varphi\rangle$ "projects $|\varphi\rangle$ onto $|\psi\rangle$" and can be thought of as picking out the portion of $|\varphi\rangle$ that overlaps with $|\psi\rangle$. If $|\varphi\rangle$ and $|\psi\rangle$ have no overlap, then $\langle\psi|\varphi\rangle = 0$, and $\hat{\mathbb{P}}_\psi|\varphi\rangle = 0$.

Projectors have the following properties:

1. Projectors have the property of **idempotency**, meaning that $\hat{\mathbb{P}}_\psi^2 = \hat{\mathbb{P}}_\psi$. Since $\hat{\mathbb{P}}_\psi^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi|$, the idempotency condition is only met if $\langle\psi|\psi\rangle = 1$, so $|\psi\rangle$ must be properly normalized for $|\psi\rangle\langle\psi|$ to be a projector onto $|\psi\rangle$.

2. If two kets $|\varphi\rangle$ and $|\psi\rangle$ are orthogonal (and therefore have no overlap), then $\langle\psi|\varphi\rangle = 0$ and the projectors onto these states are also said to be orthogonal, implying that

$$\hat{\mathbb{P}}_\psi \hat{\mathbb{P}}_\varphi = |\psi\rangle\langle\psi|\varphi\rangle\langle\varphi| = 0$$

3. A sum of orthogonal projectors is a projector onto a **subspace**. Consider a discrete basis $\{|v_k\rangle\}$ that **spans** a state space $\mathcal{E}$, where $k \in \{1, \ldots, k_{\max}\}$. For $q < k_{\max}$, the **subspace projector**

$$\hat{\mathbb{P}}_q \equiv \sum_{k=1}^{q} |v_k\rangle\langle v_k|$$

projects onto the subspace $\mathcal{E}_q$ of $\mathcal{E}$, where $\mathcal{E}_q$ is spanned by $\{|v_1\rangle, |v_2\rangle, \ldots, |v_q\rangle\}$.

Closure Relations

Every basis is associated with a **closure relation**, which is formally a projector onto the entire state space rather than onto a subspace. Every ket in the state space remains unchanged when acted upon by this projector. Consider an arbitrary state space $\mathcal{E}$. Let $\hat{\mathbb{I}}$ be the **identity operator**, such that $\hat{\mathbb{I}}|\psi\rangle = |\psi\rangle$ for any $|\psi\rangle \in \mathcal{E}$. Closure relations are expressed as follows:

Closure Relation (Discrete Basis)

For a discrete orthonormal basis $\{|v_k^i\rangle\}$, with $i \in \{1, 2, \ldots g_k\}$ and where g_k is the degree of degeneracy for index k, the closure relation is

$$\sum_k \sum_{i=1}^{g_k} |v_k^i\rangle\langle v_k^i| = \hat{\mathbb{I}}$$

Closure Relation (Continuous Basis)

For a continuous non-degenerate orthonormal basis $\{|w_\beta\rangle\}$, where β labels the (infinitely many) orthogonal elements of the basis, the closure relation is

$$\int_{\text{all }\beta} d\beta\, |w_\beta\rangle\langle w_\beta| = \hat{\mathbb{I}}$$

Because a closure relation's sum or integral is equivalent to the identity operator, it can be inserted into an expression immediately before a ket or after a bra, or next to an operator (i.e., making a product with that operator), without changing the meaning of the expression. In this manner, closure relations aid in problem solving and manipulation of expressions.

Formally, a closure relation is a mathematical statement that a basis exists and is complete; i.e, there are neither missing nor extraneous elements in the construction of the basis. The expression of a closure relation can be given as a definition of the symbols used to specify the elements of a basis.

Functions of Operators

A **function of an operator** can be defined by a **Taylor series expansion** of the same function of a continuous variable, with the operator replacing the variable. For a variable x, a function $F(x)$ has a Taylor series expansion about $x = a$ given by

$$F(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \cdot \left.\frac{d^n F(x)}{dx^n}\right|_{x=a} = \sum_{n=0}^{\infty} b_n \cdot (x-a)^n$$

where b_n is the coefficient of the n^{th}-power term. Most commonly, expansions are taken about $a = 0$. The same function of an operator $\hat{A}$ is constructed by replacing x with $\hat{A}$:

$$F(\hat{A}) = \sum_{n=0}^{\infty} b_n \cdot (\hat{A} - a)^n$$

For example, expanding e^x about $x = 0$ gives $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, so that $e^{\hat{A}} = 1 + \hat{A} + \frac{1}{2!}\hat{A}^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\hat{A}^n$. Note that $\hat{A}^0 = \hat{\mathbb{I}}$. The identity operator $\hat{\mathbb{I}}$ is often omitted from operator expansions when it is multiplied by a scalar quantity (1 in this case).

Functions of operators are themselves operators. Consider an operator $\hat{A}$ that has a non-degenerate discrete spectrum $\{\lambda_k\}$ and eigenkets $\{|v_k\rangle\}$. Two special cases for simplifying the actions of functions of operators are given below.

1. If $|v_k\rangle$ is an eigenket of $\hat{A}$ with eigenvalue λ_k, then $|v_k\rangle$ is an eigenket of $F(\hat{A})$ with eigenvalue $F(\lambda_k)$. This statement can be demonstrated by expanding $F(\hat{A})$ about 0:

$$F(\hat{A})|v_k\rangle = \sum_{n=0}^{\infty} b_n \hat{A}^n |v_k\rangle = \sum_{n=0}^{\infty} b_n \lambda_k^n |v_k\rangle = F(\lambda_k)|v_k\rangle$$

2. $F(\hat{A})$ acting on an arbitrary ket $|\psi\rangle$ can be calculated by first expanding $|\psi\rangle$ in the $\{|v_k\rangle\}$ basis: $|\psi\rangle = \sum_k c_k |v_k\rangle$, where $c_k = \langle v_k|\psi\rangle$. $F(\hat{A})|\psi\rangle$ is then evaluated using the closure relation $\hat{\mathbb{I}} = \sum_k |v_k\rangle\langle v_k|$ as follows:

$$F(\hat{A})|\psi\rangle = F(\hat{A})\hat{\mathbb{I}}|\psi\rangle = \sum_k F(\hat{A})|v_k\rangle\langle v_k|\psi\rangle = \sum_k c_k F(\lambda_k)|v_k\rangle$$