Chapter 1

Basic Concepts of the Statistical Theory of Light Scattering

1.1 Introduction

In practice, one deals mostly with objects having a continuous and, as a rule, rough surface. The surface of an object is formed under the influence of numerous random factors, such as nonideal mechanical processing, effects of temperature, and so on. As a result, the surface of an object is almost always of random shape and can generally be described by a random function of space coordinates and time. Recall that a random function of one parameter—say, time—is called a single-parameter random process.[24] Further consideration shows that random functions depend on four (three coordinates plus time) or more parameters. Such functions are called multiparameter random processes or *random fields*. This term should be used particularly for functions describing wave fields scattered by objects with random surfaces. Evidently, such fields have random structure.

It is clear that the study of wave scattering by objects with rough surfaces should be considered a statistical problem that involves finding probability characteristics of the scattered field, including distribution functions, field moments, and correlation properties, from given probability characteristics of the surface. In this chapter, we consider the basic concepts relating to the probability characteristics of random surfaces and the random fields scattered by them. The main results of the theory of diffraction by random surfaces in the Kirchhoff approximation are reviewed. Field correlation characteristics and moments scattered by moving or standing objects are considered.

1.2 Random surfaces and fields scattered by them; the Kirchhoff method

Consider the surface of an arbitrarily rough object in motion. During the course of its motion, the surface can be deformed by various forces. It is usually difficult to describe the space–time characteristics of such a surface. However, for a random surface in practice, its deviation from the mean smooth surface is small compared with the curvature radius of this surface. Such random deviations appear due to uncontrollable effects of the formation and processing of surfaces. As an example of

random deviations, one can consider the wavy surface of the sea that is formed by the influence of wind and other factors. A surface that changes its shape or position in time can be described by a radius vector that depends on the two parameters u and v, and time: $\mathbf{r} = \mathbf{r}(u, v, t)$. Random surfaces can be represented in the following form (see Fig. 1.1):

$$\mathbf{r}_\Sigma(u, v, t) = \mathbf{r}(u, v, t) + \mathbf{N}(u, v, t)\xi(u, v, t), \tag{1.1}$$

where $\mathbf{r}(u, v, t)$ is the radius vector of the mean surface, $\mathbf{N}(u, v, t)$ is the external normal to this surface at point (u, v), and $\xi(u, v, t)$ is the deviation of the random surface from the unperturbed mean surface at point (u, v) along the normal; it satisfies the condition $\langle \xi(u, v, t) \rangle = 0$. Furthermore, $\xi(u, v, t)$ will be called the surface roughness height distribution, and the mean surface is the shape of the object surface.

It is convenient to consider the parameters u and v as surface coordinates on the mean surface. We assume the corresponding coordinate frame to be orthogonal, which implies that the mean surface has certain necessary properties. In the case when the mean surface is parallel to the plane $z = 0$, we have $u = x$, $v = y$, and

$$\mathbf{r}_\Sigma(x, y, t) = \mathbf{i}x + \mathbf{j}y + \mathbf{k}[z(t) + \xi(x, y, t)],$$

where $z(t)$ describes the motion of the mean surface and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the direction of the coordinate axes.

The wave field (acoustic, electromagnetic field, etc.) scattered by a random surface is functionally related to the shape of the object, its motion, and the random function $\xi(u, v, t)$.[12] This relation can be used for studying the characteristics of

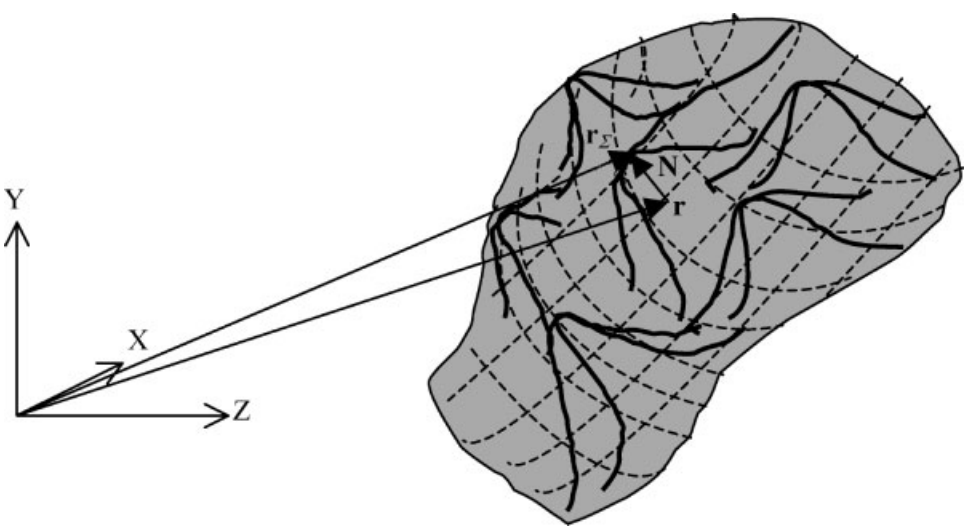

Figure 1.1 Geometry of a small part of a rough object surface. Dashed lines show the coordinates u, v on the mean surface.

the object surface and for identifying and distinguishing different objects and determining their parameters.

The advantage of such wave influence is that it does not damage the object under study and hence is a nondestructive method of control. There is a problem of determining the results of influence as a functional of the random function $\xi(u, v, t)$.[25] The result of a wave field influence is a complex process, and it is a complicated problem to describe this process for each specific realization of the surface $\xi(u, v, t)$. Usually, relatively simple analytic dependencies can be found for a field diffracted by simple structures such as a plane, smooth cylinder, smooth ball, etc., and also by periodic structures containing these simple ones. However, even diffraction of a field by gratings formed by regular flat stripes leads to a complicated field structure that has a large number of maxima and minima with different amplitudes.

In the microwave antenna technique, the field distribution in the vicinity of the main maximum is called the *main lobe*. Other maxima are called *side lobes*. Random deviations of the antenna shape from the ideal one are caused by various reasons, such as deviations of the antenna mirror surface from the ideal one (for the case of antennas with continuous opening) or irregular positions of the transmitters (for antenna arrays). Even the smallest deviation of the scattering object's position from the ideal leads to a considerable change of both amplitude and position of maxima and minima. Naturally, a field scattered by a surface with random roughness $\xi(u, v, t)$ has a complicated distribution. A complete description of field realizations is, in this case, useless in practice.

This fact has been taken into account in the antenna technique[26] and in radiolocation,[27] and a probability description of the scattered field has been developed. For instance, the values of side lobes appearing due to the deviation $\xi(u, v, t)$ of the mirror surface from the ideal one is characterized by the probability of not exceeding a given value. This statistical approach to estimating the scattered field has turned out to be quite fruitful. For instance, it allows estimation of the mean square deviation of the mirror surface from the ideal one at which the side lobes do not exceed a given value.

In the antenna technique, it is often necessary to distribute radiated energy in such a way that it is uniformly spread in space. For this purpose, the ideal mirror surface should be randomly perturbed by significant values of $\xi(\mathbf{r})$. Naturally, this problem also requires a statistical approach. Further development of this approach leads to finding such a random distribution $\xi(\mathbf{r})$ that forms a given mean distribution of the scattered field.

Hence, whenever it is necessary to study a field scattered by a random surface, of major practical interest are the statistical characteristics (mainly, mean values) of the scattered field for a group of scattering surfaces that have the same mean surface and random deviations $\xi(\mathbf{r})$ determined by identical surface processing conditions, such as grinding the surface with a fixed-size abrasive. Another example of a statistical group is a water surface. In this case, given statistical characteristics of random deviations from the mean surface are determined by the depth of the water

basin, direction and duration of the wind, and other parameters that are the same for all groups. Thus, each surface from a given group is one of the realizations described by random deviations, with fixed statistical characteristics and fixed mean surface.

The value of $\xi(u, v, t)$ at each fixed point in space–time is a random function, which can be conveniently described by the probability density $w_1(\xi, u, v, t)$. Although the function $w_1(\xi, u, v, t)$ gives the probability density distribution $\xi(u, v, t)$ at each point (u, v, t), it does not provide information about the values of the random process at neighboring points. A more detailed description of the process $\xi(u, v, t)$ is contained in the two-dimensional probability density $w_2(\xi_1, u_1, v_1, t_1, \xi_2, u_2, v_2, t_2)$ of random variables $\xi_1(u_1, v_1, t_1)$ and $\xi_2(u_2, v_2, t_2)$. Here, w_2 describes the ensemble of all functions ξ_1 and ξ_2. The most complete description of random deviations $\xi(u, v, t)$ from the mean surface is given by n-dimensional probability densities of random variables $\xi(u_1, v_1, t_1), \xi(u_2, v_2, t_2), \ldots, \xi(u_n, v_n, t_n)$:

$$w_n = w_n(\xi_1, u_1, v_1, t_1, \xi_2, u_2, v_2, t_2, \ldots, \xi_n, u_n, v_n, t_n),$$

which are connected by means of recurrent relations (matching conditions)

$$w_{n-1}(\xi_2, \ldots, \xi_n) = \int_{-\infty}^{\infty} w_n(\xi_1, \xi_2, \ldots, \xi_n) d\xi_1$$

and satisfy the normalization conditions $\int \ldots \int w_n d\xi_1 d\xi_2 \ldots d\xi_n = 1$.

Knowing the probability density w_n, one can calculate the probability densities for an arbitrary function $f(\xi_1, \xi_2, \ldots, \xi_n)$ by means of the relation

$$P(x) = \int \ldots \int \delta[x - f(\xi_1, \xi_2, \ldots, \xi_n)] w(\xi_1, \xi_2, \ldots, \xi_n) d\xi_1 d\xi_2 \ldots d\xi_n,$$

where $\delta(x - f)$ is a delta function. In particular, this leads to the equation for the mean value of a function,

$$\langle f(\xi_1, \ldots, \xi_n) \rangle = \int \ldots \int f(\xi_1, \ldots, \xi_n) \times w_n(\xi_1, u_1, v_1, t_1, \ldots, \xi_n, u_n, v_n, t_n) d\xi_1 \ldots d\xi_n. \quad (1.2)$$

Here, angle brackets $\langle\rangle$ denote averaging of the function f over the ensemble of all functions ξ.

A convenient way to describe random processes is by means of the functional of the process $\xi(u, v, t)$ probability density. This functional is defined as the limit

$$P[\xi(u, v, t)] = \lim_{n \to \infty} w_n(\xi_1, u_1, v_1, t_1, \ldots, \xi_n, u_n, v_n, t_n),$$

and, in fact, it fully determines the probability of any realization of the process $\xi(u, v, t)$. In most cases, it is not necessary to give a complete probability description of a random process. Therefore, one usually needs only simplified characteristics of the process $\xi(u, v, t)$:

the mean value of the process $\xi(u, v, t)$ (the first-order moment),

$$\langle \xi(u, v, t) \rangle = \int_{-\infty}^{\infty} \xi w_1(\xi, u, v, t) d\xi, \tag{1.3}$$

$$\psi_{12}(u_1, v_1, t_1, u_2, v_2, t_2) = \langle \xi(u_1, v_1, t_1) \xi(u_2, v_2, t_2) \rangle, \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_1 \xi_2 w_2(\xi_1, u_1, v_1, t_1, \xi_2, u_2, v_2, t_2) d\xi_1 d\xi_2; \tag{1.4}$$

the correlation function

$$B_{12}(u_1, v_1, t_1, u_2, v_2, t_2) = \psi_{12}(u_1, v_1, t_1, u_2, v_2, t_2) \\ - \langle \xi(u_1, v_1, t_1) \rangle \langle \xi(u_2, v_2, t_2) \rangle; \tag{1.5}$$

and variance of the random process

$$\sigma^2(\mathbf{r}) = \sigma^2(u, v, t) = B_{12}(u, v, t, u, v, t), \tag{1.6}$$

where σ is the standard deviation of the random process.

In practice, for any surface,

$$B_{12}(u_1, v_1, t_1, u_2, v_2, t_2) \to 0 \quad \text{at } |u_1 - u_2|, |v_1 - v_2|, t_2 - t_1 \to \infty.$$

If a random process is not statistically isotropic, it is reasonable to introduce a correlation radius in each direction, for instance, along the coordinate lines u and v on the object surface:

$$\ell_u(\mathbf{r}) = \frac{1}{\sigma^2(\mathbf{r})} \int_{-\infty}^{\infty} B_{12}(\mathbf{r}, t, \mathbf{r} + \mathbf{s}, t) r_u du$$

and

$$\ell_v(\mathbf{r}) = \frac{1}{\sigma^2(\mathbf{r})} \int_{-\infty}^{\infty} B_{12}(\mathbf{r}, t, \mathbf{r} + \mathbf{s}, t) r_v dv, \tag{1.7}$$

where $\mathbf{r} = \mathbf{r}_1$ and the vector $\mathbf{s} = \mathbf{r}_2 - \mathbf{r}_1$ has the components $s_u \approx \mathbf{r}_u(u_2 - u_1)$ and $s_v \approx \mathbf{r}_v(v_2 - v_1)$, and where $\mathbf{r}_u = (\partial x/\partial u, \partial y/\partial u, \partial z/\partial u)$ and $\mathbf{r}_v = (\partial x/\partial v, \partial y/\partial v, \partial z/\partial v)$ are two orthogonal vectors tangent to the object's mean surface at point

u_1, v_1. It is useful also to introduce the correlation time of the process $\xi(u, v, t)$:

$$\tau_c = \frac{1}{\sigma^2(\mathbf{r}, t)} \int_{-\infty}^{\infty} B_{12}(\mathbf{r}, t, \mathbf{r}, t + \tau) d\tau. \tag{1.8}$$

For statistically isotropic processes, where $\ell_u(\mathbf{r}) = \ell_v(\mathbf{r})$, one can define the correlation radius as

$$\ell(\mathbf{r}) = \ell_u(\mathbf{r}) = \ell_v(\mathbf{r}).$$

In the simplest cases, the correlation radius is an explicit parameter of the function B_{12}. For instance, in the case of an anisotropic Gaussian correlation function[12] and for such a mean surface that $\langle \xi(\mathbf{r}, t) \rangle = 0$,

$$B_{12}(u_1, v_1, t_1, u_2, v_2, t_2) = \sigma^2 \exp\left[-\frac{r_u^2 (u_1 - u_2)^2}{\ell_u^2} - \frac{r_v^2 (v_1 - v_2)^2}{\ell_v^2} - \frac{(t_1 - t_2)^2}{\tau_c^2} \right]. \tag{1.9}$$

Here, partial derivatives are taken at point u_1, v_1.

Furthermore, we assume, as a rule, that surface roughness is isotropic and the surface roughness height distribution is practically constant in time during the determination of the object parameters. In this case, $\ell(\mathbf{r}) = \ell_u(\mathbf{r}) = \ell_v(\mathbf{r})$; $\tau_c = \infty$; $\xi(u, v, t) = \xi(u, v, t_0) = \xi(\mathbf{r})$, where $\mathbf{r} = \mathbf{r}(u, v)$, and

$$\begin{aligned} B_{12}(u_1, v_1, t_1, u_2, v_2, t_2) &= B_{12}(u_1, v_1, t_0, u_2, v_2, t_0) = B_{12}(\mathbf{r}_1, \mathbf{r}_2, t_0) \\ &= \langle \xi(\mathbf{r}_1) \xi(\mathbf{r}_2) \rangle \\ &= \sigma^2 \exp\left[-\frac{r_u^2 (u_1 - u_2)^2 + r_v^2 (v_1 - v_2)^2}{\ell^2} \right], \end{aligned} \tag{1.10}$$

where t_0 is the initial moment of determining parameters of the object. Also, we will often use the notion of a *separate element of the rough surface*. This means that a part of the surface of the object under study has the area of $\sim \ell^2$. One can show that, on average, such a part contains not more than two maxima of the function $\xi(u, v)$. For instance, Fig. 1.1 shows a part of a surface that contains six separate elements of the rough surface. To avoid the account of multiple scattering, we will also assume that the surface is smooth ($\ell > 3\sigma$); and to distinguish between small changes of the object surface and the surface roughness, we will assume that $\ell < 10^3 \lambda$ and $\sigma < 10^2 \lambda$, where λ is the mean wavelength of radiation scattered by the object.

In addition to the first- and second-order moments, one can introduce higher-order moments of a random process. The mth moment is given by the relation

$$\begin{aligned} &\psi_{12\ldots m}(u_1, v_1, t_1, \ldots, u_m, v_m, t_m) \\ &\quad = \int \ldots \int \xi_1 \ldots, \xi_m w_m(\xi_1, u_1, v_1, t_1, \ldots, \xi_m, u_m, v_m, t_m) d\xi_1 \ldots d\xi_m. \end{aligned} \tag{1.11}$$

The first- and the second-order moments give a rather approximate description of a random process $\xi(\mathbf{r}, t)$. However, there exists an important class of random functions (processes) for a complete description of which it is sufficient to know these two characteristics. These are Gaussian processes. Most often, random processes describe phenomena caused by numerous independent factors. For instance, for the surface of a metal detail, deviations from the mean surface appear due to many types of mechanical processing (turning, milling, grinding, etc.). Accumulation of a large number of factors lead—according to the central limit theorem of the probability theory—to a Gaussian distribution of a random process $\xi(\mathbf{r}, t)$. For a Gaussian distribution,

$$w_1(\xi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\xi^2}{2\sigma^2}\right),$$

$$w_2(\xi_1, \xi_2) = \frac{1}{2\pi\sqrt{\sigma^4 - B_{12}^2}} \times \exp\left[-\frac{\sigma^2(\xi_1 - \langle\xi_1\rangle)^2 - 2B_{12}(\xi_1 - \langle\xi_1\rangle)(\xi_2 - \langle\xi_2\rangle) + \sigma^2(\xi_2 - \langle\xi_2\rangle)^2}{2(\sigma^4 - B_{12}^2)}\right],$$

and

$$w_n(\xi_1, \ldots, \xi_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det[B_{jk}]}} \exp\left[-\frac{1}{2}\sum_{j,k=1}^{n} W_{jk}(\xi_j - \langle\xi_j\rangle)(\xi_k - \langle\xi_k\rangle)\right], \tag{1.12}$$

where $[W_{jk}] = [B_{jk}]^{-1}$ is the matrix inverse to the correlation matrix:

$$\sum_{k=1}^{n} W_{jk}B_{km} = \delta_{jm}, \quad B_{jk} = \langle\xi_j\xi_k\rangle - \langle\xi_j\rangle\langle\xi_k\rangle. \tag{1.13}$$

Evidently, the following relation holds for the second joint moment of a random process and its derivative:

$$\left\langle[\xi(u_2, v_2, t_2)]\left[\frac{\partial\xi(u_1, v_1, t_1)}{\partial u_1}\right]\right\rangle = \frac{\partial B_{12}(u_1, v_1, t_1, u_2, v_2, t_2)}{\partial u_1}.$$

This relation follows from the linearity of averaging and differentiating.

Similarly, the following relation holds:

$$\left\langle\left[\frac{\partial\xi(u_1, v_1, t_1)}{\partial u_1}\right]\left[\frac{\partial\xi(u_2, v_2, t_2)}{\partial u_2}\right]\right\rangle = \frac{\partial^2 B_{12}(u_1, v_1, t_1, u_2, v_2, t_2)}{\partial u_1 \partial u_2}.$$

For statistically uniform processes,

$$\left\langle \frac{\partial \xi(u,v,t)}{\partial u} \right\rangle = 0,$$
$$\left\langle \frac{\partial \xi(u,v,t)}{\partial u \xi(u+\Delta u,v,t)} \right\rangle = 0.$$

It follows from these relations that the variance of a random process derivative is $B''_{xx12}(0)$. In the case of a Gaussian correlation function, the following useful relation holds:

$$\left\langle \left[\frac{\partial \xi(u,v,t)}{\partial u} \right]^2 \right\rangle = \left(\frac{\sigma}{\ell_u} \right)^2, \left\langle \left[\frac{\partial \xi(u,v,t)}{\partial v} \right]^2 \right\rangle = \left(\frac{\sigma}{\ell_v} \right)^2. \tag{1.14}$$

From the equality

$$\left\langle \frac{\partial \xi(u,v,t)}{\partial u \xi(u,v,t)} \right\rangle = \frac{\partial B_{12}(0)}{\partial u} = 0, \tag{1.15}$$

it follows that in the case of a Gaussian correlation function, the value of the random process at some point and its derivative at the same point are statistically independent. This leads to an important relation,

$$\left\langle f_1\left(\frac{\partial \xi}{\partial u} \right) f_2(\xi) \right\rangle = \left\langle f_1\left(\frac{\partial \xi}{\partial u} \right) \right\rangle \langle f_2(\xi) \rangle.$$

Let us return to the n-dimensional probability density of a Gaussian process and take the limit at n tending toward infinity. Evidently, summation in relations (1.12) and (1.13) will turn into integration according to Riemann's definition of an integral. As a result, we obtain the Gaussian probability density function (PDF), which corresponds each realization of a random process $\xi(u,v,t)$ to a certain number $P[\xi(u,v,t)]$. Equation (1.13) for the inverse correlation matrix is transformed in this case into the equation for the inverse correlation function $W(u_1, v_1, t_1, u_2, v_2, t_2)$:

$$\begin{aligned} P[\xi] &= \lim_{n\to\infty} w_n(\xi_1, \xi_2, \ldots, \xi_n) \\ &= C_f \exp\Bigg\{ -\frac{1}{2} \iint W(u_1, v_1, t_1, u_2, v_2, t_2) \\ &\qquad \times [\xi(u_1, v_1, t_1) - \langle \xi(u_1, v_1, t_1) \rangle][\xi(u_2, v_2, t_2) - \langle \xi(u_2, v_2, t_2) \rangle] \\ &\qquad \times du_1 dv_1 dt_1 du_2 dv_2 dt_2 \Bigg\}, \end{aligned} \tag{1.16}$$

where C_f is a constant and the function W satisfies the integral equation

$$\int W(u_1, v_1, t_1, u_2, v_2, t_2)B(u_2, v_2, t_2, u_3, v_3, t_3)du_2dv_2dt_2 = \delta(u_1 - u_3)\delta(v_1 - v_3)\delta(t_1 - t_3), \tag{1.17}$$

where $\delta(s)$ is a δ-function.

1.2.1 Random field scattered by an object with a random surface

Let us now consider a random field scattered by an object with a random surface. This field is formed by the summation of fields scattered by different parts of the surface. The rougher a surface and the faster its fluctuations, the larger the number of independent terms contributing to the field and, hence, the closer the field distribution function is to the Gaussian one. If the scattering surface deviates from the mean surface according to the Gaussian distribution, then its roughness and its time fluctuations are determined by four parameters: σ, ℓ_u, ℓ_v, and τ_0. In addition to the random factors mentioned above, there are certain deterministic factors that influence the scattered field, such as the shape of the mean surface, polarization of the incident wave, reflecting characteristics of the scattering surface, and the time spectrum of the radiation. A mathematical description of the scattered field that accounts for all of these factors is an extremely complicated problem. Its approximate solution can be obtained with the help of the Kirchhoff approach. Let us review its basic ideas.[3,12]

A wave source (Fig. 1.2) illuminates the surface of the object under study. The electric vector of the field $\mathbf{E}(\boldsymbol{\rho}, t)$ scattered by the object satisfies the relation

$$\mathbf{E}(\boldsymbol{\rho}, t) = \int_{-\infty}^{\infty} \mathbf{E}_0(\boldsymbol{\rho}, \omega) \exp(i\omega t)dt, \tag{1.18}$$

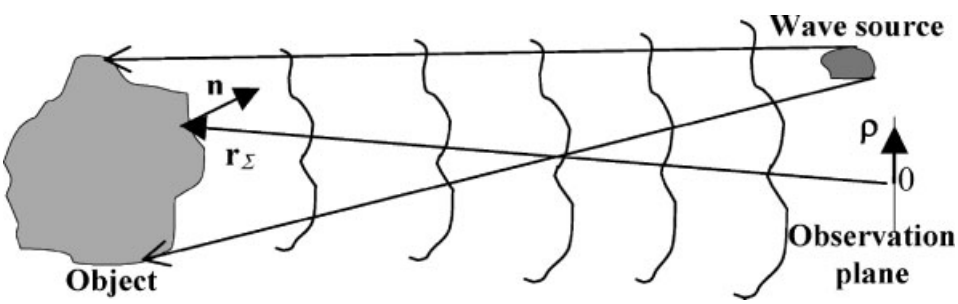

Figure 1.2 Geometry of radiation scattering by a rough object. Here, **n** is a unit vector normal to the object surface.